Homework assignment

Differentialgleichungen III
Problem Sheet 8

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http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, June 14, 2013, at 10:00am

Problem 1: Let $a > 0$. Consider the problem

$$\begin{cases} u_t = u_{xx} + u - au^3, & x \in (0, \pi), t > 0, \\ u|_{x=0} = u|_{x=\pi} = 0. \end{cases}$$

(i) Let $u(t; u_0)$ be the solution of the problem with initial condition $u|_{t=0} = u_0 \in X^{\frac{1}{2}} = \mathring{H}^1$. Show that for any $u_0 \in \mathring{H}^1$, $u(t; u_0)$ remains in a bounded region in $\mathring{H}^1$ for all $t \geq 0$.

(ii) Show that 0 is stable.

You may use the intermediate results from class.

Problem 2: Consider the problem

$$\begin{cases} u_t = u_{xx} + u, & x \in (0, \pi), t > 0, \\ u|_{x=0} = u|_{x=\pi} = 0. \end{cases}$$

Show that $u \equiv 0$ is stable but not asymptotically stable in the $\mathring{H}^1$ topology.
Hint: Use the Fourier method since the equation is linear.

Problem 3: Consider the problem

$$\begin{cases} u_t = u_{xx} - u^3, & x \in (0, \pi), t > 0, \\ u|_{x=0} = u|_{x=\pi} = 0. \end{cases}$$

Show that the solution $u \equiv 0$ is *uniformly* asymptotically stable.

Problem 4: Consider the problem

$$\begin{cases} u_t = u_{xx} + \lambda u - u^2, & x \in (0,\pi), t > 0, \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u|_{t=0} = u_0(x), \end{cases} \tag{1}$$

where λ is a positive constant. The set $C = \{u_0 \in \mathring{H}^1 : u_0(x) \geq 0 \text{ on } 0 \leq x \leq \pi\}$ is a positively invariant set for the problem, i.e. any solution of (1) with $u_0 \in C$ satisfies $u(\cdot, t) \in C$ for all $t \geq 0$ (this can be shown, using the maximum principle.)

(i) Show that the function

$$V(\varphi) = \int_0^\pi \left((\varphi')^2 - \lambda\varphi^2 + \frac{2}{3}\varphi^3 \right) dx$$

is a Liapunov function on C and that for $\varphi \in C$

$$V(\varphi) \geq (1-\lambda)||\varphi||^2 + \frac{2}{3\sqrt{\pi}}||\varphi||^3,$$

where $||\varphi|| = (\int_0^\pi \varphi^2 dx)^{\frac{1}{2}}$.

(ii) Show that (1) defines a dynamical system on C.

(iii) Assume that $\varphi \in C$ and $\dot{V}(\varphi) = 0$. Show that

(a) if $0 < \lambda \leq 1$, then $\varphi = 0$.

(b) if $\lambda > 1$, then $\varphi = 0$ or φ is the unique solution φ^+ of

$$\begin{cases} \varphi'' + \lambda\varphi - \varphi^2 = 0, & x \in (0,\pi), \\ \varphi(0) = 0, \varphi(\pi) = 0, \\ \varphi(x) > 0 & \text{on } x \in (0,\pi). \end{cases}$$

(iv) Show that

(a) if $0 < \lambda \leq 1$, then $||u(\cdot,t)||_{\mathring{H}^1} \to 0$ as $t \to \infty$.

(b) if $\lambda > 1$ and $u(x,0) \not\equiv 0$, then $||u(\cdot,t) - \varphi^+||_{\mathring{H}^1} \to 0$ as $t \to \infty$.

Hint: Examine the expression

$$\frac{d}{dt}\int_0^\pi u(x,t)\sin x dx.$$

Remark: The problem is a simple model of feedback control of a nuclear reactor. Here u is the neutron flux, which must be nonnegative.