Homework assignment

Differentialgleichungen III
Problem Sheet 9

Pavel Gurevich, Eyal Ron

http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, June 21, 2013, at 10:00am

Problem 1: Complete the details in the proof of the theorem on asymptotics of solutions near the equilibrium from the last lecture. In particular, show that

(i) $\int_0^\infty e^{\beta s} E_1 g(z(s)) ds = O(||z_0||_\alpha^{1+\delta})$,

(ii) $e^{-\beta t} \int_t^\infty e^{\beta s} E_1 g(z(s)) ds \leq \text{const}||z_0||_\alpha e^{-\gamma t}$,

(iii) $||z_2(t)||_\alpha \leq \text{const}||z_0||_\alpha e^{-\gamma t}$,

where the notation is the same as in the proof in class.

Problem 2: Let $\{T_n\}$ be a family of nonlinear operators from a Banach space X to itself. Assume that for $u \in X$

$$T_n(u) = Lu + N_n(u),$$

where L is a continuous linear operator and $||N_n(u)|| = o(||u||)$ as $u \to 0$ uniformly in $n \geq 1$. Assume that the spectral radius $r(L) := sup\{|\lambda| : \lambda \in \sigma(L)\}$ is smaller than 1.

(i) Prove that there exists $\rho > 0, M > 0, \theta < 1$, such that if $||u_0|| < \frac{\rho}{M}$ and $u_n = T_n(u_{n-1})$ for $n = 1, 2, ...$, then $||u_n|| \leq M\theta^n ||u_0||$.
Hint: If $r(L) < \nu < 1$ then there exists an equivalent norm on X, $||u||^* = \sum_{n=0}^\infty \nu^{-n} ||L^n u||$, in which $||L||^* \leq \nu$.

(ii) Define $T_n(u_0) = u(n; u_0)$ on X^α. Use this to give an alternative proof for the first linear stability theorem in the last lecture.

Problem 3: Let $Q \subset \mathbb{R}^3$ be a smooth bounded domain. Let $u = u(x,t) \in \mathbb{R}$, $v = v(x,t) \in \mathbb{R}$ and consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u - \varepsilon u f(v), & x \in Q, \\ \left.\dfrac{\partial u}{\partial \nu}\right|_{\partial Q} = 0, & \end{cases}$$

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta v + q u f(v), & x \in Q, \\ v\big|_{\partial Q} = 1, & \end{cases}$$

where D, q, ε are positive constants, ε is small and $f(v) = e^{-\frac{H}{v}}$, where H is a positive constant.

(i) Let $A : L_2(Q) \to L_2(Q)$ be the minus Laplacian operator $Au = -\Delta u$, $u \in D(A) = H^2(Q) \cap \overset{\circ}{H}{}^1(Q)$. Prove that the linearizion of the problem at $(u,v) = (0,1)$ has an eigenvalue $\lambda_0 = \varepsilon f(1)$ as a simple eigenvalue with an eigenfunction (u_0, v_0), $u_0(x) = 1, v_0(x) = (A - \lambda_0)^{-1}(qf(1))$, and all other eigenvalues in $\{Re\lambda \geq c > 0\}$ for some c independent of (small) $\varepsilon > 0$.

(ii) Prove the asymptotics

$$(u,v) = (0,1) + k(1, v_0)e^{-\lambda_0 t} + O(e^{-2\lambda_0 t})$$

as $t \to \infty$ for some constant k, provided that ε is small enough. In which normed space did you prove the last equality?