Homework assignment

Differentialgleichungen III
Problem Sheet 10

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http://dynamics.mi.fu-berlin.de/lectures/13SS-Gurevich-Dynamics/

Tutorial discussion date: Friday, June 28, 2013, at 10:00am

Problem 1: Complete the details in the proof of the theorem of linear instability from the last lecture. In particular, show that

(i) Let $a \in X_1^\alpha$, $||a||_\alpha \le \frac{\rho}{2M}$. Prove that

$$J(v)(t) := e^{-L_1(t-n)}a + \int_n^t e^{-L_1(t-s)} E_1 g(v(s))ds + \int_{-\infty}^t e^{-L_2(t-s)} E_2 g(v(s))ds, \qquad t \le n,$$

is a contraction on Ω, provided that ρ is small enough.

(ii) If $v \in \Omega$ is a solution of the integral equation $v(t) = J(v)(t), t \le n$, then it is a solution of

$$\frac{dv}{dt} + Lv = g(v), \qquad t < n.$$

(iii) $||v(n)||_\alpha \ge \dfrac{1}{2}||a||_\alpha$.

Problem 2: Consider the problem

$$\begin{cases} u_t = u_{xx} + \lambda(u - au^3), & x \in (0,\pi),\ t > 0, \\ u|_{x=0} = u|_{x=\pi} = 0, \\ u|_{t=0} = u_0(x), \end{cases}$$

where $a \in \mathbb{R}$, $\lambda \ne 1$. Prove that the zero solution is asymptotically stable if $\lambda < 1$ and unstable if $\lambda > 1$.

Problem 3: **Theorem (use without a proof):** Assume the requirements of the theorem of linear instability hold, except that the requirements on $g(z), \sigma(L)$ are replaced by:

(i) for a constant $p > 1$

$$||f(u_0 + z) - f(u_0) - Bz|| = O(||z||_\alpha^p)$$

as $z \to 0$ in X^α.

(ii) $\sigma(L) \cap \{Re\lambda < 0\}$ is nonempty.

Then the equilibrium u_0 is unstable.

Problem: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $C^\infty(\mathbb{R}^n)$ and $f(0) = 0$. Let $u = (u_1, .., u_n)^T$ and consider the problem

$$u_t = u_{xx} + f(u), \qquad x \in \mathbb{R},\ t > 0. \tag{1}$$

(i) Show that (1) defines a local dynamical system near 0 in $\mathcal{H}^1(\mathbb{R}) := \underbrace{H^1(\mathbb{R}) \times ... \times H^1(\mathbb{R})}_{\text{n times}}$.

(ii) Show that if all eigenvalues of the matrix $f'(0)$ have negative real part, then the origin is asymptotically stable, and that if at least one eigenvalue of $f'(0)$ has positive real part, then the origin is unstable.