Theoretical questions for the exam on Differential Equations III.
Summer semester 2013

Sectorial operators. Analytic semigroups. Spaces X^α

1. Sectorial operators and analytic semigroups.
    a. Definitions.
    b. **Theorem** on a complex-integral representation of analytic semigroups.
2. Negative fractional powers of sectorial operators: $A^{-\alpha}$.
    a. Definition via the integral of a semigroup.
    b. **Property**: A^{-1} = the inverse of A.
    c. **Theorem**: $A^{-\alpha}$ is bounded; $A^{-\alpha} A^{-\beta} = A^{-(\alpha+\beta)}$; representation of $A^{-\alpha}$ via the integral of the resolvent.
    d. Fourier representation of $A^{-\alpha}$ for A = "minus Laplacian in a bounded domain."
3. Positive fractional powers of sectorial operators: A^α.
    a. Basic properties.
    b. Fourier representation of A^α for A = "minus Laplacian in a bounded domain."
    c. Estimate of $A^\alpha e^{-At}$.
4. Spaces X^α.
    a. Definition.
    b. **Theorem**: $||A^\alpha u|| \approx ||B^\alpha u||$.
    c. **Properties**:
        i. X^α is well defined;
        ii. X^α is densely and continuously embedded into X^β;
        iii. X^α is compactly embedded into X^β if A has compact resolvent.
    d. Fourier representation of the elements of X^α generated by A = "minus Laplacian in a bounded domain." Embedding of X^α into the space of continuous functions on a bounded domain. Proof for the case where the bounded domain is an interval.
5. **Theorem** on a spectral decomposition (without proof).

Solvability of an abstract Cauchy problem

6. Linear homogeneous Cauchy problem.
    a. Definition of a (classical) solution.
    b. **Lemma** on existence and uniqueness of a solution (via the semigroup).
7. Linear nonhomogeneous Cauchy problem.
    a. Definition of a (classical) solution.
    b. **Lemma** on a particular solution of a nonhomogeneous problem.
    c. **Theorem** on existence and uniqueness of a classical solution. Explicit formula for the solution.
8. Semilinear Cauchy problem.
    a. Assumptions on the right-hand side. Definition of a (classical) solution.
    b. **Lemma** on the equivalence of the differential and integral equations.
    c. **Theorem** on local existence and uniqueness of solutions.
    d. **Theorem** on global existence and uniqueness of solutions (without proof).
    e. **Theorem** on compactness of solutions (without proof).
    f. **Theorem** on continuous dependence on initial data (without proof).
    g. **Theorem** on differentiability of solutions (without proof).

<u>Stability of equilibria</u>

9. Dynamical systems (nonlinear semigroups) on complete metric spaces.
    a. Definition.
    b. Motivating example: dynamical systems and autonomous semilinear Cauchy problems.
10. Equilibria of dynamical systems. Definitions:
    a. stable equilibrium,
    b. unstable equilibria,
    c. uniformly asymptotically stable equilibrium,
    d. globally asymptotically stable equilibrium.
11. Lyapunov function.
    a. Definition. **<u>Property</u>** of nonincreasing along trajectories.
    b. **<u>Theorem</u>** on the uniform asymptotic stability of the zero equilibrium for a dynamical system.
12. **<u>Example</u>** $u_t = u_{xx} + u - au^3$ with the Dirichlet boundary conditions.
    a. Abstract formulation. Local existence and uniqueness of solutions.
    b. Case $a > 0$: global asymptotic stability of $u = 0$.
    c. Case $a = 0$: stability, but no asymptotic stability of $u = 0$.
13. Linear stability of equilibria of semilinear Cauchy problems.
    a. **<u>Theorem</u>** on uniform asymptotic stability of equilibrium.
    b. **<u>Theorem</u>** on asymptotics of solutions near equilibrium.
14. Linear instability. **<u>Theorem</u>** on instability of equilibrium.
15. Saddle-point property (local stable/unstable manifolds). **<u>Theorem</u>** on existence of stable and unstable manifolds (proof for stable manifolds).
16. Chafee-Infante problem. Structure of equilibria.