Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 5: Consider the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\dot{x} \;=\; \begin{pmatrix} a & -b \\ b & a \end{pmatrix} x,$$

with $a, b \in \mathbb{R}$. Transform this linear differential equation into polar coordinates:

$$x \;=\; \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix},$$

with $r > 0$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Choose $b \neq 0$ arbitrarily and sketch phase portraits in (r, ϕ)-coordinates and in x-coordinates for $a < 0$, $a = 0$, $a > 0$.

Problem 6: Let $\Phi^{t,s} : \mathbb{R}^N \to \mathbb{R}^N$ be a *periodic* evolution with period $p > 0$, i.e.

$$\text{for all } t, s \in \mathbb{R} : \quad \Phi^{t+p,s+p} = \Phi^{t,s}.$$

Consider the *stroboscope* map $\Pi : \mathbb{R}^N \to \mathbb{R}^N$,

$$\Pi(x) = \Phi^{p,0}(x).$$

Prove:

(i) for all $k \in \mathbb{N}$: $\Phi^{kp,0} = \Pi^k$;

(ii) for each $t \in \mathbb{R}$ there exists a change of coordinates $\psi : \mathbb{R}^N \to \mathbb{R}^N$ such that for all $k \in \mathbb{Z}$: $\Phi^{t+kp,t} = \psi^{-1}\Pi^k\psi$. Determine ψ.

Problem 7: Consider the initial-value problem

$$\dot{x}(t) = x(t)^2, \qquad x(0) = x_0 = 1.$$

We know from class that the solution blows up in finite time. Discuss the discretizations

(i) explicit Euler: $x_{n+1} = x_n + \varepsilon x_n^2$,

(ii) implicit Euler: $x_{n+1} = x_n + \varepsilon x_{n+1}^2$.

In particular, calculate numerical solutions for $n \in \mathbb{N}_0$ for several (small) $\varepsilon > 0$ and compare with the exact solution. Do the discretizations explode at finite time? Why? How? Explain!

Extra credit: On which time interval do the discretized solutions converge to the exact solution for $\varepsilon \searrow 0$?

Problem 8: Let Φ^t be a flow on $X = \mathbb{R}^n$. Consider the time-shift for $\theta \in \mathbb{R}$:

$$S_\theta : \mathbb{R} \times X \to \mathbb{R} \times X, \qquad (t, x) \to (t + \theta, x)$$

on the extended phase-space $\mathbb{R} \times X$.

(i) Prove for all fixed $\theta \in \mathbb{R}$ that S_θ maps integral curves to integral curves. An *integral curve* is a subset $C \subset \mathbb{R} \times X$ which is of the form $C = \{(t, \Phi^t(x_0))\}$ for some $x_0 \in X$.

(ii) When is an integral curve C fixed under some S_θ? When is it fixed under all S_θ?