Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 25: Let the vector field $f = f(x)$ on $\mathbb{R}^N$ be globally Lipschitz continuous with Lipschitz-constant L. For $\lambda \in \mathbb{R}$, consider the set $Y_\lambda \subseteq C^0(\mathbb{R}, \mathbb{R}^N)$ for which

$$\|x\|_\lambda = \sup_{t\in\mathbb{R}} |x(t)| \exp(-\lambda t)$$

is finite. Note that Y_λ is a Banach-space with the norm $\|\cdot\|_\lambda$.
Prove that the Picard-Lindelöf iteration becomes a contraction on Y_λ for suitable λ. How are λ and L related?

Problem 26: The initial-value problem

$$\dot{x} = f(x) := x^2, \qquad x(0) = x_0 := 1$$

has a solution for $-\infty < t < 1$ with "blow-up", $\lim_{t\to 1} x(t) = +\infty$.
Let $(x_k)_{k\in\mathbb{N}}$ be the series of Picard iterates:

$$\begin{aligned} x_0(t) &\equiv x_0, \\ x_{n+1}(t) &= x_0 + \int_0^t f(x_n(s))\, \mathrm{d}s. \end{aligned}$$

(i) Prove: $x_k(t)$ is defined for all $k \in \mathbb{N}$ and $t \in \mathbb{R}$.

(ii) Calculate $x_1(t)$, $x_2(t)$ and $x_3(t)$ explicitly.

(iii) Determine all $t \geq 0$ such that $x_k(t)$ converges to the solution $x(t)$ of the initial-value problem, as $k \to \infty$.

Problem 27: Consider a continuously differentiable vector field $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$. Let $x(t, t_0)$ denote the solution at time t of the associated initial-value problem

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0.$$

Prove: For any fixed τ such that $x(\tau, t_0)$ exists, $x(\tau, t_0)$ is differentiable with respect to t_0. Which differential equation is satisfied by the partial derivative $v(t) := \partial_{t_0} x(t, t_0)$?

Problem 28: Let f be the vectorfield of the flow Φ^t. Prove or disprove:

The linearization $\mathrm{D}_{x_0}\Phi^t(x_0)$ of the flow, at an equilibrium x_0, is the flow of the linearized vector field $\mathrm{D}_{x_0}f(x_0)$.