Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

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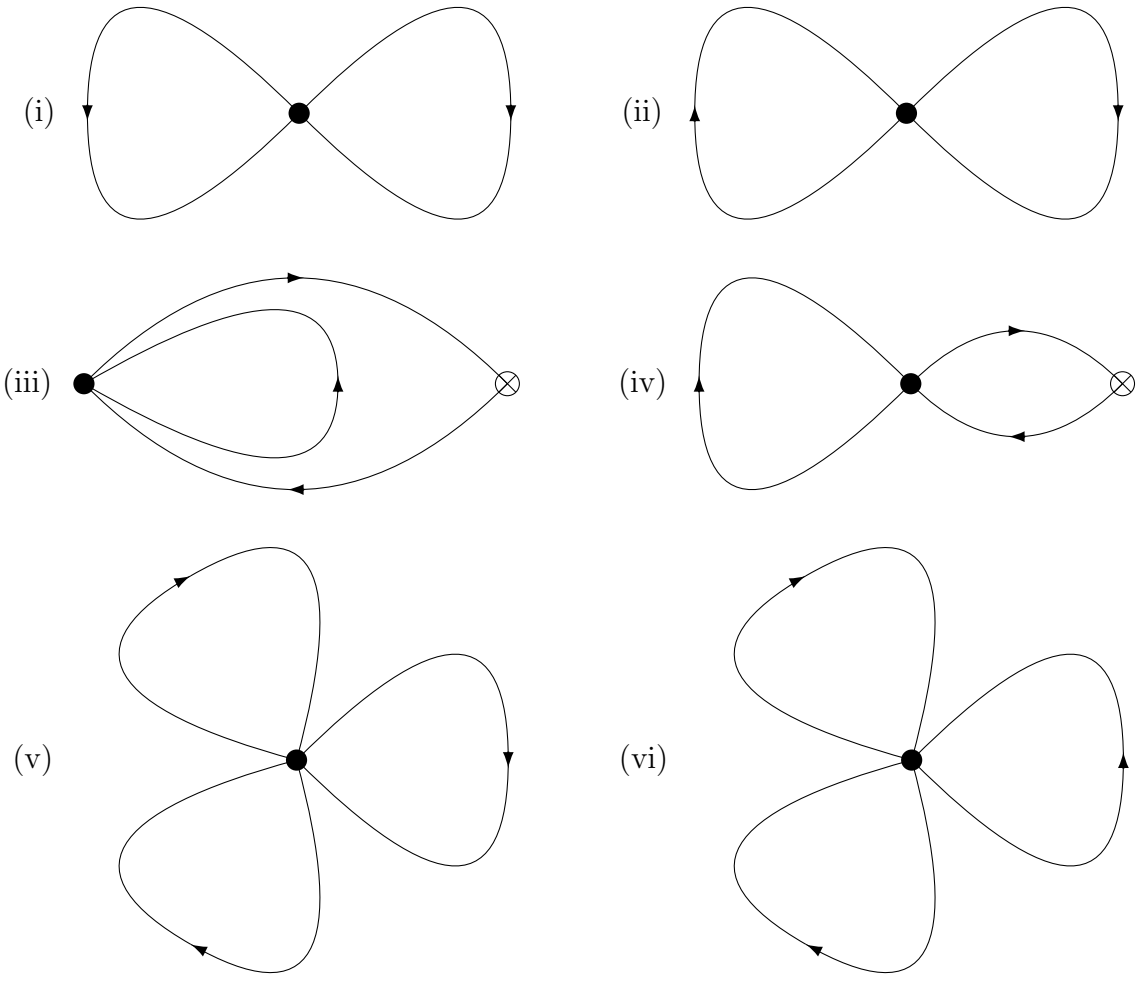
This last set of four problems is not counted as a course requirement. Solutions, however, will be credited as usual and can therefore compensate some lacking points.

Problem 45:

Which of the following sets are possible ω-limits of a single trajectory of some planar flow? Which of the sets cannot occur as ω-limits of a single trajectory? Justify your claims, without providing explicit vector fields.



Discs ● denote equilibria (of any type) and crossed out circles ⊗ denote hyperbolic saddles.

Problem 46: Prove or disprove the theorem of Poincaré & Bendixson for flows on

(i) the sphere S^2,

(ii) the torus T^2.

Problem 47: Consider a continuous flow on X and a non-empty, compact, and invariant subset $M \subset X$.

Prove or disprove: M is stable if, and only if, every open neighborhood of M contains a positively invariant open neighborhood of M.

Recall: An open neighborhood of M in X is any open subset of X which contains $\mathrm{clos}(M)$.

Problem 48: [Arnol'd, Moscow exams] It is known since Ibn Sahl and Ibn al-Haytham that the sines of the angles formed by the incident and refracted rays with the normal to an interface of different media are inversely proportional to the indices of refraction of the media:

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{n_2}{n_1}.$$

a) Find a differential equation for the light rays in the plane $(x, y) \in \mathbb{R}^2$, if the index of refraction is $n = n(y)$.

b) Study the case $n(y) = 1/y$.

Remark: Taking light rays as straight lines, the half plane $\{y > 0\}$ with the index of refraction $n(y) = 1/y$ becomes a model of Lobachevsky non-Euclidean geometry.

c) Alternatively to (b), draw the rays emanating in different directions from the origin in a plane with index of refraction $n = n(y) = y^4 - y^2 + 1$.

Remark: This explains the formation of a mirage (fata morgana): the air over the desert attains its maximal density and refraction at a certain finite height. This is due to thinner air in higher and hotter lower layers.

Acoustic channels in the ocean are an analogous phenomenon: maximal refraction occurs at a depth of 500-1000m.