Dynamical Systems II

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Only attempted exercises will be discussed. The exercises are for extra credit, i.e. points you obtain will count towards your “Übungsschein”, however we will not use these questions to calculate the threshold necessary to obtain the “Übungsschein”.

Weihnachtsaufgabe 1: Find a flow φ_t on the 2-torus with (two or more) periodic orbits — but without equilibria — such that no global transverse section S exists.

Weihnachtsaufgabe 2: Let $\mu > 0$ be fixed. Consider a map $v : [0,1] \to [0,1]$ and the vertical cone $S^+ = \{\, (\xi, \eta) \,:\, |\xi| \le \mu|\eta| \,\}$.

(i) Prove that v is a horizontal curve (i.e. v is Lipschitz continuous with Lipschitz constant μ) if, and only if, its graph lies inside every vertical cone attached to it (i.e. $\operatorname{graph}(v) \subset S^+ + (v(y), y)$ for all $y \in [0,1]$).

(ii) Let v be differentiable. Prove that v is a horizontal curve if, and only if, every tangent vector lies in S^+.

(iii) Formulate similar cone conditions for maps $v : \mathbb{R}^n \to \mathbb{R}^n$.

Weihnachtsaufgabe 3: Let $\Phi : M \to M$ be a continuous map on a metric space M. We call a sequence $(\xi_k)_{k\in\mathbb{N}}$ a δ-*pseudo orbit*, if the estimate

$$\operatorname{dist}(\Phi(\xi_k), \xi_{k+1}) \;<\; \delta.$$

holds for all $k \in \mathbb{N}$. We call a Φ-orbit $(x_k)_{k\in\mathbb{N}} = (\Phi^k(x_0))_{k\in\mathbb{N}}$ in M an ε-*shadow* of the pseudo orbit $(\xi_k)_{k\in\mathbb{N}}$ if the estimate

$$\operatorname{dist}(x_k, \xi_k) \;<\; \varepsilon$$

holds for all $k \in \mathbb{N}$. We say that the pair (M, Φ) has the *shadow property*, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that every δ-pseudo orbit has an ε-shadow.

(i) Prove: the shift on two symbols has the shadow property.

(ii) Give an interpretation of the shadow property from a numerical view point.

Weihnachtsaufgabe 4: Consider again the bouncing-ball map $f_{\alpha,\gamma}$ on $S^1 \times \mathbb{R}$:

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma \cos(\Phi_j + v_j), \end{aligned}$$

with $0 < \alpha < 1$ and $0 < \gamma$. Define the domain

$$D := \left\{ (\Phi, v) \in S^1 \times \mathbb{R} \; : \; |v| \le \frac{\gamma}{1-\alpha} + \varepsilon \right\}$$

for some $\varepsilon > 0$. Prove:

(i) D is positively invariant, i.e. $f_{\alpha,\gamma}(D) \subseteq D$.

(ii) D is absorbing, i.e. for all (Φ_0, v_0) there exists $n_0 \in \mathbb{N}$ such that $(\Phi_n, v_n) \in D$ for all $n \ge n_0$.

(iii) The *global attractor*, defined by

$$\mathcal{A}_{\alpha,\gamma} := \bigcap_{n=0}^{\infty} f^n_{\alpha,\gamma}(D),$$

is compact and invariant under $f_{\alpha,\gamma}$ as well as $f^{-1}_{\alpha,\gamma}$. A compact invariant set is maximal if it contains every compact invariant set. Furthermore is $\mathcal{A}_{\alpha,\gamma}$ the *maximal* compact and invariant set.

(iv) $\mathcal{A}_{\alpha,\gamma}$ is indeed attracting, i.e. for all (Φ_0, v_0)

$$\lim_{n\to\infty} \operatorname{dist}\Big((\Phi_n, v_n), \mathcal{A}_{\alpha,\gamma} \Big) = 0.$$

(v) $\mathcal{A}_{\alpha,\gamma}$ contains the closure of the set of all periodic points of $f_{\alpha,\gamma}$.

Weihnachtsaufgabe 5: Consider the iteration on the 2-torus $T = (\mathbb{R}/\mathbb{Z})^2$ the matrix

$$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Find a horseshoe for a suitable iterate B^k, $k > 0$.

Hint: Identify the torus with the unit square centered at $(0,0)$ and investigate the images of a parallelogram parallel to the eigenvectors of B.

Weihnachtsaufgabe 6: Consider the bouncing-ball map f,

$$\begin{aligned} \Phi_{j+1} &= \Phi_j + v_j, \\ v_{j+1} &= \alpha\, v_j - \gamma \cos(\Phi_j + v_j), \end{aligned}$$

discussed in class, with $\alpha = 1$ and $\gamma \gg 0$. Find a “horseshoe” giving rise to an invariant set I such that $f|I$ is conjugate to the shift on 6 symbols.

Extra credit: Find a shift on m symbols for every $m \ge 2$.

Weihnachtsaufgabe 7: [Horocycles] Consider the POINCARÉ-model of hyperbolic geometry, i.e., the upper half plane $\mathcal{H} = \{z = (x, y) \in \mathbb{R}^2 : y > 0\}$ with the arclength element

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The geodesics (i.e. locally shortest paths) of $\mathcal{H}$ are the vertical straight lines, $\{z = (x, y) : x = x_0, y > 0\}$, and the (Euclidean) circles with centers on the x-axis, $\{z = (x, y) : (x - x_0)^2 + y^2 = r^2, y > 0\}$, $c \in \mathbb{R}$, $r > 0$.

Consider the geodesic flow Φ on the unit tangent bundle $T^1\mathcal{H}$. Trajectories of Φ are given by geodesics of $\mathcal{H}$ with attached unit tangent vectors.

(i) Show that $(0, t)$ is a unit vector in $T_{(0,t)}\mathcal{H}$ and thus the curve $(z(t), \dot{z}(t)) = ((0, e^t), (0, e^t)) \in T^1\mathcal{H}$ is a trajectory of the geodesic flow.

(ii) Consider horizontal lines $W^s(t) := \{((x, e^t), (0, e^t)) \in T^1\mathcal{H}; x \in \mathbb{R}\}$. Show that horizontal lines are mapped onto horizontal lines under the geodesic flow. Show that horizontal lines are stable leaves (Blätter/fibers/manifolds), i.e. starting with $(w, \dot{w}) \in W^s(0)$ we have $\mathrm{dist}_{\mathcal{H}}(z(t), w(t)) \to 0$ as $t \to \infty$.

(iii) Prove that the inversion $\sigma : \mathcal{H} \to \mathcal{H}$,

$$\sigma(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right),$$

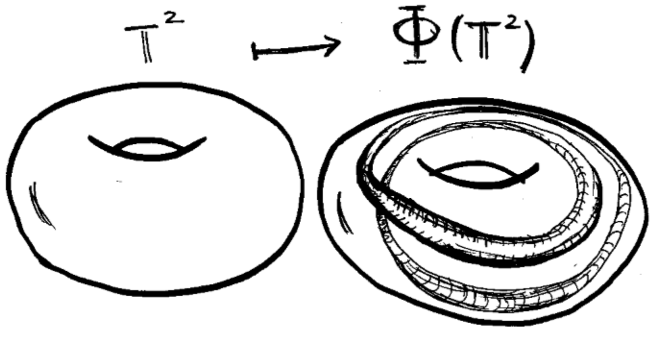
in the unit circle of $\mathbb{R}^2$ defines an isometry of $\mathcal{H}$. Thus σ maps trajectories $z(t)$ of the geodesic flow onto trajectories and reverses time. Use this to prove that the unstable leaves $W^u(t)$ of the trajectory (i) are given by circles. Sketch the families $W^u(t)$ and $W^s(t)$.

(iv) Horizontal translations

$$\tau_a(x, y) = (x + a, y)$$

are also isometries of $\mathcal{H}$. Choose $a \neq 0$ and sketch or plot the image of $z(t)$, $W^u(t)$, $W^s(t)$ under the isometry $\sigma \circ \tau_a$.

Weihnachtsaufgabe 8: Consider the self-map Φ of the (solid) 2-Torus $\mathbb{T}^2 := \text{disk} \times S^1$ as depicted in the figure. See also Problem 2 on Sheet 9. Consider the Smale solenoid $\mathcal{A} := \bigcap_{n=0}^{\infty} \Phi^n(\mathbb{T}^2)$.



Prove or disprove:

(i) The Smale attractor is connected.

(ii) The Smale attractor is path-connected.

Weihnachtsaufgabe 9: [Subshift of finite type] Let $\Sigma_N = \{0, 1, \dots, N-1\}^{\mathbb{Z}}$ be the set of 2-sided sequences on N symbols and $\sigma : \Sigma_N \to \Sigma_N, (x_k)_{k\in\mathbb{Z}} \mapsto (x_{k+1})_{k\in\mathbb{Z}}$ the shift.

Consider a matrix $A = (a_{k\ell})_{0\leq k,\ell\leq N-1} \in \{0,1\}^{N\times N}$ with entries 0 or 1. Let every row and every column of A contain at least one nonzero entry. Define the set

$$\Sigma_A := \left\{ x = (x_k)_{k\in\mathbb{Z}} \in \Sigma_N \;\middle|\; a_{x_k x_{k+1}} = 1 \text{ for all } k \in \mathbb{Z} \right\}$$

of sequences respecting the transfer matrix A. (The transfer matrix A determines valid successors x_{k+1} of elements x_k in sequences $x \in \Sigma_A$.)

Note that Σ_A is nonempty and invariant under the shift σ. The shift σ on the set Σ_A is also called subshift of finite type.

(i) Let

$$\Sigma_{A,n,\alpha,\beta} := \{ (x_0, x_1, \dots, x_n) \mid x \in \Sigma_A,\ x_0 = \alpha,\ x_n = \beta \}$$

be the set of finite sequences of length $n+1$ which start with α and end with β. Prove that the number of elements of $\Sigma_{A,n,\alpha,\beta}$ is given by the corresponding entry of A^n, i.e.

$$|\Sigma_{A,n,\alpha,\beta}| = (A^n)_{\alpha\beta}.$$

(ii) Prove that the topological entropy (see Problem 4 on Sheet 9) is determined by the largest eigenvalue of A, i.e.

$$h_{\text{top}} = \log |\lambda_{\max}(A)|.$$

Weihnachtsaufgabe 10: Consider again the subshift of finite type, i.e. the shift σ on the set of sequences

$$\Sigma_A := \left\{ x = (x_k)_{k\in\mathbb{Z}} \in \Sigma_N \;\middle|\; a_{x_k x_{k+1}} = 1 \text{ for all } k \in \mathbb{Z} \right\}$$

for a given transfer matrix $A = (a_{k\ell})_{0\le k,\ell\le N-1} \in \{0,1\}^{N\times N}$ with entries 0 or 1. Let every row and every column of A contain at least one nonzero entry. Assume furthermore, that for every $0 \le k, \ell \le N-1$ there exists a positive integer n, such that the matrix entry $(A^n)_{k,\ell}$ is nonzero.

(i) Prove that σ possesses a dense orbit in Σ_A.

(ii) Describe the set of periodic orbits. Is it countable?

Weihnachtsaufgabe 11: Let Φ^s be a flow on a metric space X. Consider the set of nonwandering points. A nonwandering point $x \in X$ is a point such that every neighborhood U of x there exists a $T > 1$ with $U \cap \phi^T(U) \neq \emptyset$. Prove or disprove:

(i) The set of nonwandering points is invariant.

(ii) The set of nonwandering points is closed.

Weihnachtsaufgabe 12: Calculate the Taylor expansion of the stable and unstable manifold of $(0,0)$ for the following system differential equations

$$\dot{x} = -x - y^2, \qquad \dot{y} = y + x^2$$

up to order 5