White exam

Dynamical Systems III

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Problem 1: Consider the equation

$$\dot{x} = f(\mu, x)\ \mu \in \mathbb{R},\ x \in \mathbb{R}\ f \in C^r, r \geq 2.$$

(i) What are the conditions for a saddle-node, a transcritical, a pitchfork bifurcation to take place at (μ_0, x_0)?

(ii) Give the normal forms of the bifurcations above and draw their bifurcation diagrams with stabilities of the branches.

Problem 2: Consider the system of ODE's with $(x, y) \in \mathbb{R}^2$, $\mu \in \mathbb{R}$.

$$\dot{x} = (3+\mu)x + (3-\mu)y - (x-y)^2, \tag{1}$$
$$\dot{y} = (3-\mu)x + (3+\mu)y \tag{2}$$

(i) Determine the equilibria of the system.

(ii) Find (μ_0, x_0, y_0) for which a bifurcation takes place, determine the type of bifurcation and draw the bifurcation diagram near (μ_0, x_0, y_0) with stability of the branches.

Problem 3: Consider the second order ODE, $x \in \mathbb{R}$, $\mu \in \mathbb{R}$

$$\ddot{x} + (\mu - 3x^2)\dot{x} + x - x^3 = 0$$

(i) Write this equation as a first order system and determine all the equilibria.

(ii) Which equilibria undergo bifurcation as μ varies? Determine the type of bifurcations taking place (with local bifurcation diagram and stability of the branches) or the stability of the equilibria without bifurcation.

Problem 4: Consider an ODE

$$\dot{x} = f(\mu, x),$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, f smooth. Assume that there is an equilibrium at (μ_0, x_0) satisfying all three conditions

$$\begin{array}{lll} \text{(A)} & \frac{\partial f}{\partial x}(\mu_0, x_0) & = 0 \\ \text{(B)} & \frac{\partial f}{\partial \mu}(\mu_0, x_0) & \neq 0 \\ \text{(C)} & \frac{\partial^2 f}{\partial x^2}(\mu_0, x_0) & = 0, \quad \dfrac{\partial^3 f}{\partial x^3}(\mu_0, x_0) \neq 0 \end{array}$$

(i) Give a simple example of an ODE satisfying conditions (A), (B), and (C) at $(\mu_0, x_0) = (0, 0)$ and draw its bifurcation diagram. Do you observe branching of equilibria? Changes of stability?

(ii) For a general f satisfying conditions (A), (B), and (C) , show that there is a curve of equilibria going through (μ_0, x_0) in the (μ, x)-plane and show that the bifurcation diagram of the first question describes the general Situation. For that, find out how the signs of $\mu - \mu_0$ and $\frac{\partial f}{\partial x}(\mu, x)$ change along the curve of equilibria.

stability formula

In a two dimensional system of the form

$$\begin{cases} \dot{x} & = -\omega y + f(x, y) \\ \dot{y} & = \omega x + g(x, y) \end{cases}$$

where $f(0, 0) = g(0, 0) = 0$ and $Df(0, 0) = Dg(0, 0) = 0$, the coefficient a of the Hopf normal form responsible for the stability is given by:

$$a = \frac{1}{16}(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}) + \frac{1}{16\omega}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy})$$