White exam

Perturbation Theory

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Problem 1: Consider the initial value problem, with $\varepsilon > 0,\ x \in \mathbb{R}^n\ f \in C^r, r \geq 2$

$$\begin{cases} \dot{x} & = \varepsilon f(t, x, \varepsilon), \\ x(0) & = x_0 \end{cases}$$

where f is T-periodic in the time variable t, and $\bar{f}(y) = \frac{1}{T}\int_0^T f(t, y, 0)$ its average. Let $D \subseteq \mathbb{R}^n$. Furthermore assume that f and its derivative are bounded $\mathbb{R} \times D \times [0, \varepsilon_0]$, and that trajectories with initial conditions in D do not leave int$D$ on time scale $1/\varepsilon$.

(i) Determine $u(t, y, \varepsilon)$, $f_1(t, y, \varepsilon)$ such that $y = x - \varepsilon u$ satifies the initial value problem

$$\begin{cases} \dot{y} & = \varepsilon \bar{f}(y) + \varepsilon^2 f_1(t, y, \varepsilon) + O(\varepsilon^3), \\ y(0) & = x_0. \end{cases}$$

(ii) Assume that $f(t, y, \varepsilon) = f(t, y)$ Prove that u is bounded and periodic in t.

(iii) State the Gronwall lemma and prove that the solution $z(t)$ of

$$\begin{cases} \dot{z} & = \bar{f}(z) \\ z(0) & = x_0 \end{cases}$$

satisfies $|y(t) - z(t)| = O(\varepsilon)$ on a time scale $1/\varepsilon$.

(iv) State the first order averaging Theorem, periodic case.

Problem 2: Consider the differential equation

$$\dot{x} = \varepsilon \left\{ \sin(x + t) + \varepsilon x \sin^2 t \right\},$$

where $x \in \mathbb{R}$, $\varepsilon > 0$ and small. Prove the existence of small periodic orbits near $x = 1$ and determine their stability.

Problem 3: Consider ODE with $x \in \mathbb{R}$, $\varepsilon > 0$

$$\dot{x} = \varepsilon \exp(-t)x$$

(i) Solve the equation explicitely, give the asymptotics for $|t|$ large and sketch the graph of $x(t)$ for an initial condition $x_0 > 0$. Compute $\sup_{t\geq 0} |x_0 - x(t)|$.

(ii) Apply the Averaging Theorem (nonperiodic case) for x in a Ball centered at the origin and compare the estimate it produces to the estimate computed for the explicit solutions.