J-PD – Basic Questions

1. What is the definition of partial differential equations? What is its order and when do we speak of quasilinear and semilinear equations?

2. What do we call the characteristic of a first order differential equation and in what sense do they "solve" the equation?

3. How can the scalar, $x \in \mathbb{R}$, wave equation $u_{tt} = u_{xx}$ with initial conditions $u(0,x) = u_0(x)$ and $u_t(0,x) = u_1(x)$ be solved by a characteristic ansatz?

4. Consider the scalar wave equation, $x \in \mathbb{R}$, $u_{tt} = u_{xx}$ with initial conditions $u(0,x) = u_0(x)$ and $u_t(0,x) = u_1(x)$. What can you say about the dependence of the solutions on the initial conditions and about the regularity of the solutions with respect to x?

5. Consider the heat equation $u_t = \Delta u$, $x \in \mathbb{R}^n$, with initial condition $u(0,x) = u_0(x)$. What can you say about the regularity of solutions with respect to x?

6. Define linear, strongly continuous semigroups and their infinitesimal generators on a Banach space X and give 2 examples including the space X.

7. Let $T(t)$ be any strongly continuous semigroup on a Banach space X. Is the map $\begin{array}{c} [0,\infty) \times X \longrightarrow X \\ (t,u) \longmapsto T(t)u \end{array}$ continuous? Prove or disprove.

8. Let $T(t)$ be a strongly continuous semigroup on a Banach space X, prove or disprove the existence of constants such that $T(t)$ is controllable with respect to $\|\cdot\|_{\mathcal{L}(X)}$.

11. How is the domain $D(A) \subseteq X^{\text{Banach}}$ of an operator defined? Define when ~~a~~ an operator is called closed, when densely defined? Give an example of a closed, densely defined operator, its domain as well as the Banach ~~s~~ space X in which it lies.

12. Define what the resolvent set $\rho(A)$ of an operator A is and what its relation is to the spectrum of A.
Give an example of an operator, its resolvent set and its spectrum.

13. Assume $T(t)$ is ~~an~~ linear strongly continuous semigroup and the operator A is defined by $D(A) = \{u \in X \mid Au := \lim_{t \searrow 0} \frac{1}{t}(T(t)u - u) \text{ exists}\}$.
What conclusions can u draw about A, $\frac{d}{dt} T(t)u$ and especially about the resolvent?

14. In which case does an operator A on a Banach space X coincide with the infinitesimal generator of a linear strongly continuous semigroup $T(t)$. Formulate a theorem.

15. Define what a weak derivative is and prove its uniqueness.

16. Formulate the Lax-Milgram-theorem.
What conclusions can you make regarding the solvability of PDE's? Give an easy example.

17. Consider the partial differential equation

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega \end{cases}$$

with $u \in C^2(\Omega) \cap C^1(\partial\Omega)$, $\partial\Omega$ smooth, $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and ~~is~~ $\frac{\partial u}{\partial \nu} = \nu^T \nabla u$ where ν is the ~~unit~~ outward normal on $\partial\Omega$.
Derive a weak formulation of the PDE and give a suitable solution and test spaces for it.

18. Define in which case we call an operator A dissipative and maximal dissipative. Further formulate the Lumer-Phillips theorem.

19. How can we define $\exp(At)$ for an infinitesimal generator A and for which operators are these definitions good and for which are they bad? Further what is the so called Yosida regularization?

21. Consider the general inhomogenous linear differential equation
$$\begin{cases} u'(t) = Au(t) + f(t), & t \geq 0 \text{ and } f \in C^0([0,T], X), \\ u(0) = u_0 \end{cases}$$
and the variation-of-constants solution of ODE's
$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s) ds,$$
where $T(t)$ denotes the strongly continuous semigroup of the infinitesimal generator A.
When do we say, u is a strong, when a mild solution? Further proof that any strong solution is also a mild solution and hence unique.

22. Consider again $\begin{cases} u'(t) = Au(t) + f(t), \; t \geq 0 \text{ and } f \in C^0([0,T], X), \\ u(0) = u_0 \end{cases}$

and
$$u(t) = T(t)u_0 + \int_0^t T(t-s) f(s) ds,$$
where $T(t) = \exp(At)$.
Under which assumptions on u_0 and f is a mild solution strong?

23. Define "sector" $S_{\beta,\omega}$, analytic semigroups as well as when we call an operator sectorial.

24. Let $A \in \mathcal{H}(M, \beta, \omega)$. Under which assumptions is A the infinitesimal generator of an analytic semigroup? What further conclusions can you draw?

25. Define fractional powers $(-A)^\alpha$ for operator $A \in \mathcal{H}(M, \beta, \vartheta)$ and the Banach space X^α with its norm $|\cdot|_\alpha$

26. Let $A \in \mathcal{H}(M, \beta, \vartheta)$. Assume $\beta < 0$, $\|(\lambda - A)^{-1}\| \leq \frac{M}{1+|\lambda|}$.
What conclusions can you make regarding $(-A)^\alpha T(t)$, X^α, $(-A)^\alpha T(t)$ on X^α, the controllability of $T(t)u$ with respect to $|\cdot|_\alpha$ and the local continuity of $T(t)u$?

27. Consider (1) $u'(t) = Au + F(u)$ with $(u(0) = u_0$ and
$$(2)\ u(t) = (\Phi(u))(t) := T(t)u_0 + \int_0^t T(t-s)F(u(s))ds.$$
We assume $F: X^\alpha \to X$, $0<\alpha<1$, is locally Lipschitz, i.e.
(Lip)$_\alpha$ $\forall u_0 \in X^\alpha\ \exists V \subset X^\alpha$, s.t. $u_0 \in V$, and Lipschitz constant L, s.t. $\forall u, v \in V: \|F(u) - F(v)\|_X \leq L \cdot |u - v|_{X^\alpha}$.
Formulate the abstract theorem about the existence and uniqueness of solutions of (1) and (2).

28. Consider
$$(1)\ u'(t) = Au + F(u),\ u(0) = u_0$$
$$(2)\ u(t) = (\Phi(u))(t) := T(t)u_0 + \int_0^t T(t-s)F(u(s))ds$$
with $F: X^\alpha \to X$, $0<\alpha<1$, locally Lipschitz.
Remember the abstract theorem about the existence and uniqueness of solutions of (1) and (2).
Does $F(t,u)$ work as well? Under which further assumptions?
Can global solutions be obtained? Under what conditions?
If we assume A^{-1} is compact and a solution $\{u(t) | t \geq 0\}$ is global and uniformally bounded. What can you say about the ω-limit set $\omega(u_0) := \{v = \lim u(t_n)$ for some seq. $t_n \to \infty\}$?

29. What is a Sobolev exponent of a Sobolev space $W^{k,p}(\Omega)$ and what is its importance regarding the embeddings of $W^{k,p}(\Omega)$ into $C^{k,\alpha}(\Omega)$ and $W^{k,p}(\Omega)$ respectively? Formulate a theorem.

30. Consider $Lu := \sum_{j,k} \partial_{x_k}(a_{jk}(x)\partial_{x_j}u) + \sum_j b_j(x)\partial_{x_j}u + c(x)$ on $\Omega \subseteq \mathbb{R}^n$ with $\partial\Omega$ in C^2, $a, b, c \in L^\infty$, $c \leq -\lambda$ negative enough and L elliptic, i.e. $\xi^T(a_{jk})\xi \geq \delta|\xi|_2^2 > 0$.
Formulate a theorem about the embeddings of X^α into $W^{k,p}$ and $C^{0,\eta}$?