Infinite-Dimensional Dynamics

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Assignment 1
due date: Monday, October 26, 2015, 5 p.m.

Problem 1: Consider the one-dimensional wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad (t, x) \in [0, \infty) \times \mathbb{R}.$$

Show that the general solution $u \in C^2$ takes the form

$$u(t, x) = f(x + ct) + g(x - ct),$$

with $f, g \in C^2$. How are f and g related to the initial conditions for u, u_t?

Problem 2: Let $u \in C^3$ be a positive solution of the heat equation

$$u_t = \mu\, u_{xx},$$

where $\mu > 0$ is a constant.

(i) Show that $\vartheta := -2\mu\, u_x/u$ solves Burgers equation:

$$\vartheta_t + \vartheta\vartheta_x = \mu\, \vartheta_{xx}.$$

(ii) Reconstruct u from ϑ. Which solution u do you obtain if you put $\vartheta = x/t$?

Problem 3: We call $u : \mathbb{R}^n \to \mathbb{R}$ homogeneous function of degree $\alpha > 0$ if

$$u(\lambda x) = \lambda^\alpha u(x) \qquad \text{for all } x = (x_1, \dots, x_n) \in \mathbb{R}^n \text{ and for all } \lambda > 0.$$

Prove or disprove the following statement for $u \in C^1$:

$$u \text{ is homogeneous of degree } \alpha, \text{ if and only if } x \cdot \nabla_x u = \alpha u.$$

Problem 4: Consider a three-dimensional stationary and incompressible fluid with zero viscosity. Show that the Bernoulli equation

$$\tfrac{1}{2}|u|^2 + p = \text{const}\,.$$

holds for C^1-solutions u of the inviscid Navier-Stokes equation.