Infinite-Dimensional Dynamics

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Assignment 2
due date: Monday, November 2, 2015, 5 p.m.

Problem 5: For $u \in L^2(\mathbb{R})$ we call $\hat{u} \in L^2(\mathbb{R})$ the Fourier transform,

$$\hat{u}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} u(x)\,\mathrm{d}x,$$

$||u||_{L^2(\mathbb{R})} = ||\hat{u}||_{L^2(\mathbb{R})}$. Let $u \in C^2$ be a solution of the heat equation $u_t = u_{xx}$ such that $u_{xx}(t,\cdot) \in L^2(\mathbb{R})$ for each fixed $t > 0$.

Which ordinary differential equation is fulfilled by the Fourier transform with respect to x, $\widehat{u(t,\cdot)}(k)$? Which solution $u(t,x)$, $t > 0$, do you obtain for the initial condition $\hat{u}(k) \equiv 1$?

Problem 6: Prove that we can express a solution $u(t,x)$, $t > 0$, $x \in \mathbb{R}^N$, $u \in C^\infty((0,\infty) \times \mathbb{R}^N)$, of the heat equation $u_t = \triangle u$ in the form

$$u(t,x) = \big(G(t,\cdot) * u_0(\cdot)\big)(x) = \int_{\mathbb{R}^N} G(t, x-y)\, u_0(y)\,\mathrm{d}y$$

where

$$G(t,\xi) = \frac{1}{(2\sqrt{\pi t})^N} \exp\left(-\frac{|\xi|_2^2}{4t}\right),$$

and $u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Extra credit: Does the initial condition $u(0,x)$ coincide with $u_0(x)$?

Problem 7: Consider the wave equation $u_{tt} = c^2 u_{xx}$ on the interval $(-1,1)$ for $t > 0$ and constant wave speed c. We choose *transparent boundary conditions*: $u_t \pm c u_x = 0$ at $x = \pm 1$.

Show that all solutions $u \in C^2$ are independent of t and x for any $t > 2/c$.

Can you give a physical interpretation of the boundary conditions and the constant solution for $t > 2/c$?

Problem 8: Let $u(t)$ be the solution of the delay differential equation

$$\dot{u}(t) = f(u(t), u(t-1))$$

with given prehistory $u_0(\vartheta)$, $-1 \leq \vartheta \leq 0$, $u_0 \in C^0$. Suppose $f \in C^\infty$.

Prove: $u \in C^k$ for $t \in [k-1, T)$, $k = 0, 1, 2, \ldots$, where T is the maximal interval of existence of the solution.