Infinite-Dimensional Dynamics

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Assignment 3
due date: Monday, November 9, 2015, 5 p.m.

Problem 9: Consider the shift $(T(t)u_0)(x) := u_0(x+t)$. Prove or disprove the following statements:

(i) $T(t)$ is a semigroup on $X = L^\infty(0,\infty)$.

(ii) $T(t)$ is a strongly continuous semigroup on $X = L^\infty(0,\infty)$.

(iii) $T(t)$ is a strongly continuous semigroup on $X = BC^0(0,\infty)$ (bounded and continuous functions with the sup-norm).

(iv) $T(t)$ is a strongly continuous semigroup on $X = BC^1(0,\infty)$ (subset of $BC^0(0,\infty)$ of functions with bounded and continuous first derivative with the sup-norm for the functions and their derivatives).

(v) $T(t)$ is a strongly continuous semigroup on $X = BC^0_{\text{unif}}(0,\infty)$ (bounded and uniformly continuous functions with the sup-norm).

Problem 10: Consider the operator $A = \mathrm{d}^2/\mathrm{d}x^2$ on the space $X = C^0([0,1])$ of continuous functions. The domain of A is given by

$$\mathcal{D}(A) = \{u \in C^2([0,1]) \mid u(0) = u(1) = 0\}.$$

Is A a closed operator?

Consider the same operator A, but with Neumann boundary conditions of the interval $[0,1]$. Is A a closed operator?

Extra credit. Is A a closed operator on $BC^0(\mathbb{R})$, without any boundary conditions?

Problem 11: Consider identical mathematical pendula at points $x \in h\mathbb{Z}$ along the x-axis. Each pendulum at $x = nh$, $n \in \mathbb{Z}$, moves in the yz-plane and is coupled to its two nearest neighbours at $x = (n-1)h$ and at $x = (n+1)h$ with a linear torsion spring of strength $1/h^2$. For the angle u_{nh} of the pendulum at $x = nh$, we find the following equation:

$$u''_{nh}(t) = \sin(u_{nh}(t)) + \frac{1}{h^2}\left(u_{(n-1)h} - 2u_{nh} + u_{(n+1)h}\right).$$

Which partial differential equation can be obtained in the formal limit $h \searrow 0$?

Let w be the winding number of the pendula around the x-axis for $-\infty < x < \infty$. Find non-constant travelling waves with $w = 0, 1, \infty$.

Problem 12: Let $a, b \in BC^1(\mathbb{R}, \mathbb{R})$ be positive functions. Consider a solution $u \in C^1(\mathbb{R}^2, \mathbb{R})$ of

$$a(x)\partial_x u + b(y)\partial_y u = 0.$$

Prove the existence of functions $f, g, h \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$u(x, y) = f(g(x) + h(y)).$$

Hint. Make an ansatz for f, g, h and prove the identity using characteristics.