Infinite-Dimensional Dynamics

Bernold Fiedler, Isabelle Schneider

http://dynamics.mi.fu-berlin.de/lectures/

Assignment 5

due date: Monday, November 23, 2015, 5 p.m.

Problem 17: Motivated by the delay equation $\dot{x}(t) = x(t-1)$ we define recursively

$$x(t) := x(N) + \int_{N-1}^{t-1} x(s)\,\mathrm{d}s, \qquad \text{for } N < t \leq N+1,\ N = 0, 1, 2, \ldots$$

with given prehistory $x(\cdot) = x_0(\cdot) \in X := C^0([-1,0], \mathbb{R})$. Now let

$$(T(t)x_0)(\vartheta) := x(t+\vartheta), \qquad \text{for } t \geq 0,\ -1 \leq \vartheta \leq 0.$$

Prove that $T(t)$ is a strongly continuous semigroup. What is the infinitesimal generator and its domain?

Problem 18: Let $(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y$ be the convolution and $T(t)$ the semigroup of the linear heat equation on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Prove

(i) $\|f * g\|_{L^p} \leq \|f\|_{L^1}\|g\|_{L^p}$ for $1 \leq p < \infty$.
Hint: Try $p = 1$ first. Then decompose $|f(x-y)| = |f(x-y)|^{1/q}|f(x-y)|^{1/p}$, for $p > 1$, and apply the Hölder inequality.

(ii) $\|T(t)\|_{L^p} \leq 1$, i.e. $T(t)$ is a contracting semigroup.

Problem 19: Consider the damped harmonic oscillator

$$\ddot{x} + \alpha\dot{x} + x = 0,$$

where $x \in \mathbb{R}$, and $\alpha \in \mathbb{R}$ is the constant damping parameter. Consider the three cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$. When is the system dissipative?

Compare the parameters α for which the system is dissipative to those parameters α for which the system loses energy. The energy is given by

$$E(x, \dot{x}) = \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}x^2,$$

i.e. the sum of kinetic and potential energy.

Problem 20: Show that the explicit formula

$$u(t,x) = \frac{1}{2}\big(u_0(x-t) + u_0(x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} u_1(s)\,\mathrm{d}s$$

defines a semigroup for $(u_0, u_1) := (u, u_t) \in X := H^1(\mathbb{R}) \times L^2(\mathbb{R})$. How is the infinitesimal generator related to the linear wave equation?