Infinite-Dimensional Dynamics

Bernold Fiedler, Isabelle Schneider
http://dynamics.mi.fu-berlin.de/lectures/

Assignment 6
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Problem 21: Let Ω be a bounded domain in $\mathbb{R}^n$, with boundary $\partial\Omega$ of class C^1, and $h : \partial\Omega \to \mathbb{R}$ continuous. Prove that for $u, v \in H^1(\Omega) = W^{1,2}(\Omega)$

$$B[u, v] := \int_\Omega (\nabla u \cdot \nabla v + kuv) + \int_{\partial\Omega} huv$$

is bounded and coercive as long as the real constant $k > 0$ is chosen large enough.

Let $f \in L^2$. Which partial differential equation is satisfied by a (sufficiently smooth) solution u of

$$B[u, v] = \int_\Omega f \cdot v \, \mathrm{d}x, \qquad \forall v \in H^1(\Omega)\,?$$

In particular, discuss the boundary conditions.

Problem 22: We are interested in the product and chain rules for Sobolev functions on bounded domains Ω and Ω':

(i) For $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$ with $1/p + 1/q = 1$, prove that the product $fg \in W^{1,1}(\Omega)$ and that $(fg)' = f'g + g'f$ holds for the weak derivatives.

(ii) Let $\tau : \Omega' \to \Omega$ be a C^1-diffeomorphism with uniformly bounded derivatives of τ and τ^{-1}. For $f \in W^{1,p}(\Omega)$ prove that $f \circ \tau \in W^{1,p}(\Omega')$ and that the chain rule is valid.

Problem 23: Transform the wave equation $u_{tt} = u_{xx}$, with $x \in \mathbb{R}$, $t \geq 0$, to a symmetric hyperbolic system for $\underline{u} = (p, v, u) \in L_2(\mathbb{R})^3$. Here $p := u_x$ and $v := u_t$.

Assume that the initial conditions for p and u_x coincide, i.e. $p(0, x) = u_x(0, x)$. Does $p(0, x) = u_x(0, x)$ hold for all times $t \geq 0$ (and hence $u \in H^1(\mathbb{R})$)?

What is the infinitesimal generator? Find an explicit expression for the solution of the semi-group of the symmetric hyperbolic system.

Problem 24: Let A be a densely defined and closed operator on the Hilbert space H. Assume that all positive (real) λ are contained in the resolvent set of A with the estimate:

$$\|(A - \lambda)^{-1}\| \leq \lambda^{-1}, \qquad \forall\, \lambda \in (0, \infty).$$

Prove that A is dissipative and even maximal dissipative.

Hint: Consider the scalar product $\big((A - \lambda)u, (A - \lambda)u\big)_H$.

Extra: Find an operator A that is not dissipative but has a resolvent set which contains all positive λ (of course without the above estimate).