Infinite-Dimensional Dynamics

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Assignment 7
due date: Monday, December 7, 2015, 5 p.m.

Problem 25: A strongly continuous family of bounded operators $T(t)$, $t \in \mathbb{R}$, on a Banach space X is called (strongly continuous) *group* if $T(t)u$ is continuous in t, $T(0) = \text{id}$, and $T(t+s) = T(t)T(s)$ for all real t, s (not only positive), $u \in X$.

Let A be the infinitesimal generator of the semigroup $T(t)$.

(i) Recall that A is closed and densely defined. Prove that there exist constants M, β such that
$$|(A-\lambda)^{-n}| \leq \frac{M}{(|\text{Re}(\lambda)| - \beta)^n}$$
for all $|\text{Re}(\lambda)| > \beta$ and $n = 1, 2, 3, \ldots$.

(ii) Prove that a given operator A generates a (strongly continuous) group $T(t)$ if (i) is satisfied.

(iii) Find an *interesting* example of a (strongly continuous) group $T(t)$.

Extra credit: How about the case $t \in \mathbb{C}$?

Problem 26: We consider the shift $(T(t)u)(x) := u(t+x)$ on $X = L^p(\mathbb{R}^+)$, $1 \leq p < \infty$, $t \geq 0$. Let A be the infinitesimal generator. Prove that the resolvent set of A consists of all $\lambda \in \mathbb{C}$ with strictly positive real part.

Hint: Discuss $(A - \lambda)u$ for $u(x) = \exp(\lambda x)$ and use the fact that the resolvent set is open.

Problem 27: Consider strongly continuous semigroups T, S on X with infinitesimal generators A, B, respectively. Assume $T(t)S(\tau) = S(\tau)T(t)$ for all $t, \tau > 0$.

Prove that $T(t)S(t)$ is a strongly continuous semigroup. What is its infinitesimal generator and domain?

Problem 28: Let $A : \mathcal{D}(A) \subseteq X \to X$ be densely defined in the Banach space X. Prove that $\|u\|_1 := \|u\|_X + \|Au\|_X$ defines a norm on $\mathcal{D}(A)$. Prove that $\mathcal{D}(A)$ with this norm is a Banach space if, and only if, A is closed.