Infinite-Dimensional Dynamics

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Assignment 8
due date: Monday, December 14, 2015, 5 p.m.

Problem 29: Using a Fourier ansatz, prove a resolvent estimate for the operator

$$A : \mathcal{D}(A) = H^2(S^1) \subseteq L^2(S^1) \longrightarrow L^2(S^1), \qquad u \longmapsto u_{xx}$$

on the interval $(0, 2\pi)$ with periodic boundary conditions $u(t,0) = u(t,2\pi)$, $u_x(t,0) = u_x(t,2\pi)$ for $t \geq 0$. Conclude that A generates a strongly continuous semigroup.

Problem 30: Consider the operator

$$A : \mathcal{D}(A) = (H^2 \times H^1)(\mathbb{R}) \subset (H^1 \times L^2)(\mathbb{R}) \longrightarrow (H^1 \times L^2)(\mathbb{R}), \qquad (u,v) \longmapsto (v, -u_{xx}).$$

Prove that A is densely defined and closed but does not generate a strongly continuous semigroup.

Problem 31: Let $T(t)$ be a strongly continuous semigroup of bounded operators. Assume that for every $t > 0$, $T(t)^{-1}$ exists and is a bounded operator.

Show that $S(t) := T(t)^{-1}$ is also a strongly continuous semigroup of bounded operators with infinitesimal generator $-A$.

Conclude that

$$U(t) := \begin{cases} T(t) & \text{for} \quad t \geq 0 \\ T(t)^{-1} & \text{for} \quad t \leq 0 \end{cases}$$

is a strongly continuous group of bounded operators.

Problem 32: Let $u(t,x)$, $t > 0$, $x \in \mathbb{R}$, be defined as

$$u(t,x) := \begin{cases} -1 & \text{for} \quad x \leq -t \\ 0 & \text{for} \quad |x| < t \\ +1 & \text{for} \quad x \geq t. \end{cases}$$

Show that u is a weak solution of the wave equation, i.e. for all smooth v with compact support in $\{t > 0, x \in \mathbb{R}\}$ we have

$$0 = \iint u\,(v_{tt} - v_{xx})\,\mathrm{d}x\,\mathrm{d}t.$$

Extra credit: Can you extend the weak solution to negative t?