Infinite-Dimensional Dynamics
Bernold Fiedler, Isabelle Schneider
`http://dynamics.mi.fu-berlin.de/lectures/`

Assignment 10
due date: Monday, January 25, 2016, 5 p.m.

Problem 37: [Henry §1, ex. 8] Let A be sectorial and $T(t)$ the generated analytic semigroup on X. Prove inductively for every number $m \in \mathbb{N}$:

(i) $R(T(t)) \subset D(A^m)$ for all $t > 0$.

(ii) $D(A^m)$ is dense in X.

Problem 38: [Henry §1, ex. 9] Let $A \in G(M, 0)$ be the generator of the strongly continuous semigroup $T(t)$ on X. Assume additionally $R(T(t)) \subseteq D(A)$ and $\|AT(t)\| \le M/t$, for all $0 \le t \le 1$. Prove:

(i) $\forall t \in (0, 1],\ m \in \mathbb{N} \qquad \|(\mathrm{d}/\mathrm{d}t)^m T(t)\| = \|A^m T(t)\| \le M^m m^m t^{-m}$.

(ii) $T(t)$, $t \ge 0$ is an analytic semigroup.

Problem 39: Consider the scalar delay differential equation

$$\dot{u}(t) = -u(t-1) \tag{1}$$

with given prehistory $u_0(\vartheta)$, $-1 \le \vartheta \le 0$, $u_0 \in C^0$.

(i) What is the corresponding infinitesimal generator A?

(ii) Note $\lambda \notin \rho(A)$ if $u(t) := \exp(\lambda t)$ solves (1). The resulting equation $\chi(\lambda) := \lambda + \exp(-\lambda) = 0$ is called the *characteristic equation*.

(iii) Show that A is not sectorial.
Hint. Show that the eigenvalues $\lambda = \mu + i\nu$, $\lambda \in \mathbb{C}$, $\mu, \nu \in \mathbb{R}$ do not lie in a sector. It is useful to split the characteristic equation into real and imaginary parts and find expressions $\mu(\nu) = \ldots$ and $\nu(\mu) = \ldots$.

Extra credit: Even though the infinitesimal generator is not sectorial, we can show the following: If there is a sequence $\chi(\lambda_n) = 0$ such that $|\lambda_n| \to \infty$ as $n \to \infty$, then $\operatorname{Re}\lambda_n \to -\infty$ as $n \to \infty$. Conclude the following: Any halfplane given by $\operatorname{Re}\lambda > a$ for $a \in \mathbb{R}$ contains at most finitely many eigenvalues.

Problem 40: Give another example of a strongly continuous but non-analytic semigroup and its generator. Prove directly (without using the theorem) that the generator is not sectorial.