Infinite-Dimensional Dynamics

Bernold Fiedler, Isabelle Schneider
`http://dynamics.mi.fu-berlin.de/lectures/`

Assignment 11
due date: Monday, February 1, 2016, 5 p.m.

Problem 41: Let $A \in \mathcal{H}(M, \beta, \delta)$ be sectorial, $\beta < 0$, $0 < \alpha < 1$. Prove:

$$(-A)^{-\alpha} \;=\; \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty t^{-\alpha}(t-A)^{-1}\,\mathrm{d}t$$

Why can't you use the formula for $\alpha \geq 1$?

Extra credit (until February 8): For $0 < \alpha < n+1$, $\alpha \notin \mathbb{Z}$, show that

$$(-A)^{-\alpha} \;=\; \frac{\sin(\pi\alpha)}{\pi} \cdot \frac{n!}{(1-\alpha)\cdots(n-\alpha)} \int_0^\infty t^{n-\alpha}(t-A)^{-(n+1)}\,\mathrm{d}t$$

(see [H. Komatsu], Pacific J. Math **19** (1966)).

Problem 42: Consider the operator $Au = u_{xx}$ on the space $X = L^2(S^1)$, i.e. with periodic boundary conditions, $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Prove that A is sectorial and describe the space X^α using conditions on the Fourier coefficients $u_k \in \mathbb{C}$ of the series

$$u(x) \;=\; \sum_{k=0}^{\infty} \mathrm{Re}\left(u_k \mathrm{e}^{\mathrm{i}kx}\right).$$

Problem 43: Let $A \in \mathcal{H}(M, \beta, \delta)$ be a sectorial operator on X, $\beta < 0$. Prove that

$$T(\alpha) := (-A)^{-\alpha}, \qquad \alpha \geq 0$$

is a strongly continuous (even analytic) semigroup on $\mathcal{L}(X, X)$. What is the infinitesimal generator?

Problem 44: Let $A \in \mathcal{H}(M, \beta, \delta)$ be a sectorial operator of the analytic semigroup $T(t)$ on X, $\beta < 0$. Assume $f \in C^0((0,\tau], X^\alpha)$, $0 < \alpha \leq 1$, and that $\|(-A)^\alpha f(t)\|$ is bounded. Prove that the mild solution of

$$(*) \qquad\qquad u'(t) = Au(t) + f(t)$$

is a strong solution of $(*)$ for all initial conditions $u(0) = u_0 \in X$.