Infinite-Dimensional Dynamics

Bernold Fiedler, Isabelle Schneider

http://dynamics.mi.fu-berlin.de/lectures/

Voluntary problems

due date: Monday, January 11, 2016, 5 p.m.

Only attempted exercises will be discussed.
You will obtain extra credit points for solved exercises.

Problem X1: For integrable $f: [0, 2\pi] \to \mathbb{C}$ we define the Fourier coefficients as

$$f_k := \frac{1}{2\pi} \int_0^{2\pi} f(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x, \qquad k \in \mathbb{Z}.$$

Show

(i) $f \in L^2 \iff \sum |f_k|^2 < \infty$;

(ii) $f \in H^1 \iff \sum k^2 |f_k|^2 < \infty$.

Find analogous descriptions for H_0^1, H^2 and H_0^2.

Problem X2: The partial differential equation

$$u_t = u_{xx} + \sum_{j=1}^{n} \sin(jx) f_j(\Phi_1[u], \ldots, \Phi_n[u]), \qquad 0 < x < \pi$$

with boundary conditions $u = 0$ in $x = 0, \pi$ and

$$\Phi_j[u] := \int_0^{\pi} u(t, x) \sin(jx)\, \mathrm{d}x$$

has solutions of the form

$$u(t, x) = \sum_{j=1}^{\infty} u_j(t) \sin(jx).$$

(i) Which ordinary differential equations do the $u_j(t)$ fulfill?

(ii) Assume that the f_j are bounded. How does the solution $u(t, \cdot)$ behave in the limit $t \to +\infty$?

Problem X3: Solve the wave equation

$$u_{tt} - u_{xx} = 0, \qquad (t,x) \in (0,\infty) \times (0,1)$$

with boundary conditions $u(t,0) = u(t,1) = 0$. Use the ansatz $u(t,x) = \varphi(t)\psi(x)$ (separation of variables) and expand $x \mapsto u(t,x)$ as a Fourier series in x with coefficients depending on t. What is the solution if we prescribe initial conditions $u_t(0,x) \equiv 0$ and

(i) $u(0,x) = \sin(\pi x)$,

(ii) $u(0,x) = \sum_{k=1}^{\infty} a_k \sin(k\pi x)$ or

(iii) $u(0,x) = 1 - 2|x - \frac{1}{2}|$?

Explore the differentiability properties of $u(t,\cdot)$, as t evolves.

Problem X4: Consider the wave equation (summation convention)

$$v_{tt} + \alpha v_t = b_{jk} v_{x_j x_k} + c_j v_{x_j} + dv, \qquad x \in \mathbb{R}^n$$

with v real, b symmetric and uniformly positive definite, α, c, d in BC^0, and b in BC^1. Which regularity of the coefficients is needed to define a semigroup for (v, v_t) in $H^{k+1} \times H^k$? Which k provides classical solutions (that means all needed derivatives are continuous) in $\mathbb{R}^3$?

Problem X5: Let $A \in G(M,\beta)$ be the generator of a strongly continuous semigroup $T(t)$ on X. Define $X_n := D(A^n)$, $n = 0,1,2,\ldots$, with the graph norm $\|u\|_n := |A^n u|_X + |u|_X$ for $n \in \mathbb{N}$. Prove:

(i) $D(A^{n+1}) \subset X^n$ is dense.

(ii) The operator $A_n := A|_{X_n}$ with $D(A_n) = D(A^{n+1})$ generates the restricted semigroup $T_n(t) = T(t)|_{X_n}$ and satisfies $A_n \in G(M,\beta)$.

Problem X6: Let A satisfy the assumptions of the Lumer-Phillips theorem on the Banach space X, i.e. A is dissipative and $R(\lambda_0 - A) = X$ for some real $\lambda_0 > 0$. Which of the following operators generate a strongly continuous semigroup (proof!), and which do not (counterexample!)?

(i) A^2

(ii) $-A^2$

(iii) $\exp(At)$

(iv) $\exp(-At)$

Problem X7: Let $u \in C^2((0,\infty) \times (0,1)) \cap C^0([0,\infty) \times [0,1])$ be a solution of the heat equation

$$u_t = u_{xx}, \qquad x \in (0,1),$$

with $u(t,0) = u(t,1) = 0$ and $u(0,x) = \varphi(x)$.

Show that $u \in C^\infty((0,\infty) \times (0,1))$.

Define solutions also for $\varphi(x) \in L^2([0,1])$! Do the solutions become smooth?

Hint: Calculate the Fourier coefficients of the solution and its x-derivatives.

Problem X8: Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be linear. Define

$$T(t)u := \exp(tA)u = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}.$$

Show that $T(t)$ defines a strongly continuous semigroup. What is its infinitesimal generator?

Problem X9: Let X be a Banach space and $T(t)$, $t \geq 0$ a strongly continuous semigroup with infinitesimal generator A. Suppose that λ is an eigenvalue of A with corresponding eigenfunction u_λ.

Show that

$$T(t)u_\lambda = e^{\lambda t} u_\lambda$$

holds for all $t \geq 0$.

Problem X10: Consider the energy functional

$$\begin{aligned} I:\ H^1(\Omega) &\longrightarrow \mathbb{R} \\ u &\longmapsto \int_\Omega (|\nabla u|^2 + ku^2 - f \cdot u)\,\mathrm{d}x, \end{aligned}$$

where $f \in L^2(\Omega)$, $k \in (0,\infty)$ are given and Ω is a bounded domain. Prove:

(i) I is continuously differentiable.

(ii) I has exactly one critical point u^*, i.e. $I'(u^*) = 0$.

(iii) $I(u^*) = \min_{u \in H^1(\Omega)} I(u)$.

Which partial differential equation does u fulfill if you assume that u and $\partial\Omega$ are smooth?

Problem X11: Anna-Liz tells Analyx about some formula like

$$f(\lambda) = \frac{1}{2\pi i} \oint f(z)(z-\lambda)^{-1}\,\mathrm{d}z,$$

for any complex analytic function f, i.e., f is complex differentiable at every point $\lambda \in \mathbb{C}$. Here the integral is evaluated along a positively (i.e., left) oriented closed Jordan curve Γ with λ in the interior. Now Analyx wonders what the integrals

$$\frac{1}{2\pi i} \oint (z-A)^{-1} dz, \qquad \frac{1}{2\pi i} \oint e^{tz}(z-A)^{-1}\,\mathrm{d}z$$

might be, for a *matrix* A. Can you help him? Try Jordan curves which enclose

(i) only one simple eigenvalue λ of A,

(ii) only one multiple eigenvalue λ,

(iii) all eigenvalues.

Problem X12: Consider the equation

$$u_t(t) = Au(t) + f(t)$$

on the Banach space X with $u(0) = u_0$, $A \in G(M, \beta)$, and continuous f. Assume $\beta < 0$ and the existence of the limit

$$f_\infty := \lim_{t \nearrow \infty} f(t).$$

Prove the existence of an asymptotic limit of the mild solution u:

$$u_\infty := \lim_{t \nearrow \infty} u(t).$$

Which equation is solved by u_∞?

Problem X13: Solve

$$u(t) = \int_0^t k(s)u(t-s)ds + f(t), \quad t \geq 0 \tag{1}$$

by Laplace transform, assuming $f, k \in BC^0$ and $0 \leq k(t) \leq C^{-\alpha t}$ for some constants $\alpha, C > 0$.

Hint. The Laplace transform $G(\lambda)$ of a function $g(t)$ is defined by

$$G(\lambda) := \int_0^\infty e^{-\lambda t} g(t)\,\mathrm{d}t.$$

Apply Laplace transform to both sides of (1) and solve the algebraic equation which you obtain.

Problem X14: View (1) in Problem X13 as a semigroup

$$u'(t) = Au(t) + f'(t), \quad t \geq 0.$$

We consider the prehistories

$$(u(t))(\vartheta) := u(t+\vartheta), \quad \infty < \vartheta \leq 0$$

and rewrite (1) as

$$u(t) = \int_{-\infty}^{t} k(t-s)u(s)\,\mathrm{d}s.$$

Which Banach spaces, domains, and definitions would you recommend for A? What is your resolvent estimate?

Problem X15: Compare the answers of Problem X13 and Problem X14.

Merry Christmas and a Happy New Year!