Homework assignment

Infinite-Dimensional Dynamical Systems

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Problem 9: Let K be a compact subset of a metric space X. Let $n^X(\varepsilon, K)$ count the minimum number of closed ε - balls with centers in X needed to cover K. Define the "X-capacity"

$$c^X(K) := \overline{\lim} \frac{\log n^X(\varepsilon, K)}{\log 1/\varepsilon},$$

for $\varepsilon \searrow 0$. Show $c^X(K) = c^K(K)$.

Problem 10: Let Y be a finite-dimensional subspace of a real Banach space X. Assume $\xi \in X \setminus Y$ with $|\xi| = 1$.

(i) Show that there exists $y \in Y$ with $|\xi - y| = \text{dist}(\xi, Y)$. Is y necessarily unique?

(ii) Determine $\sup |y|$ in (i), over all possible triples X, Y, ξ as above. At least try $\dim X = 2$.

Problem 11: Let K be a compact subset of a Banach space X. Assume $f \in C^1(U, X)$ in a neighborhood U of K, and $f(K) \supseteq K$. Let $K_p := \bigcap_{0 \leq j \leq p} f^{-j}(K)$, for $p \in \mathbb{N}_0$. Show $f^p(K_p) \supseteq K \supseteq K_p$ and $c(K_p) = c(K)$.

Problem 12: Let the equivalence relation $X \sim Y$ denote that the two Banach spaces X, Y are isometrically linearly isomorphic. Let $Q(N)$ denote the set of equivalence classes of real N-dimensional Banach spaces, under $\sim$. On $Q(N)$ define

$$d(X, Y) := \log \inf \|T\| \left\|T^{-1}\right\|,$$

where inf is taken over all invertible linear $T : X \to Y$.

(i) Show, that d is a well-defined metric on $Q(N)$.

(ii) Consider $N = 2$. Let the unit circle in X be the square, and the unit circle in Y the regular hexagon. What is the distance d of their equivalence classes in $Q(2)$?

Note: $Q(N)$ is called the Banach-Mazur compactum. In fact $\text{diam}(Q(2)) = \log 3/2$.