Homework assignment

Infinite-Dimensional Dynamical Systems

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Problem 37: Determine the spectrum of the Laplace-Beltrami operator Δ_* on the unit sphere $S^2 := \{(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta);\ 0 \le \vartheta \le \pi,\ 0 \le \varphi < 2\pi\}$, i.e,

$$\Delta_* u(\vartheta,\varphi) = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u(\vartheta,\varphi)}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 u(\vartheta,\varphi)}{\partial^2\varphi}.$$

Hint: Eigenfunctions are given by the restriction to S^2 of homogenous polynomials $h = h(x,y,z)$ such that $\Delta h = 0$ holds for the usual Laplacian Δ.

Problem 38: Consider the equation $u_t = \Delta u + f(u)$, $(x,y) \in [-1,1]^2$ with Dirichlet boundary conditions, $f \in C^1$ and $uf(u) < 0$ for large $|u|$. Show that the system is equivariant under the reflection $T((x,y)) = (-x,y)$, i.e. $T\Delta = \Delta T$ and $T(f(u)) = f(T(u))$. Conclude that there exists an equivariant inertial manifold Φ, i.e. define the restriction of T to P, Q by T_1, T_2. Then holds

$$T_2(\Phi(p)) = \Phi(T_1(p)).$$

Prove that if $p(t)$ is a solution of the reduced equation, then also $T_1 p(t)$.

Problem 39: Consider the equilibria $u(t,x) = v(x)$ of the PDE

$$u_t = u_{xx} + \lambda u(1-u^2)$$

with Neumann boundary conditions on the interval $[0,\pi]$ with real parameter λ.

(i) Why are the bifurcation points from the trivial equilibrium $u \equiv 0$ located at $\lambda = 0, 1, 4, 9, ...$? Are the bifurcations sub- or supercritical, i.e. does λ decrease or increase along the bifurcating, non-trivial local bifurcation branches?

(ii) Show that each branch is globally parametrized over $v(0)$, i.e, $\lambda = \lambda(v(0))$, for $|v(0)| < 1$ and determine sign $\lambda'(v(0))$.

Problem 40: Let $f \in BC^1$ and assume $\varepsilon > 0$ is sufficiently small. Construct an infinite-dimensional (!) manifold which contains all classical solutions $u = u(x, y) \in BC^2$ of the elliptic equation

$$\begin{aligned} \Delta u &= \varepsilon f(u), \\ u(x,0) &= u(x,1) = 0, \end{aligned}$$

for all $x \geq 0$ and $0 < y < 1$.