2. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, May 6, 2016

Exercise 6. Calculate the general solution of $x'(t) = \sqrt{x(t)}$ by "separated variables". Why does this not provide all solutions? What is the set of all solutions of this equation? (6 points)

Exercise 7. Calculate the solution of the following problems to an arbitrary initial value condititon $x(t_0) = x_0$ and determine the maximal interval of existence of the solution.

a) $x' = t^2x^2$. (2 points)
b) $x' = t^2/x$. (2 points)
c) $x' = t^2$. (2 points)
d) $x' = x^2$. (2 points)

Exercise 8. Find the general solution of the following ODEs

a) $x' = t^2x + t^3e^{t^3/3}$. (2 points)
b) $x' = \frac{x}{1+t^2} + 2t - 1$. (2 points)
c) $x' = x/t + t^3$. (2 points)

Exercise 9. For $\alpha \neq 1$, the so-called *Bernoulli equation* has the form

$$\dot{x} = g(t)x + h(t)x^\alpha$$

(for integer α, the power is declared in the obvious way also for negative x.)

a) Assuming $x(t) \neq 0$, use the substitution $y = x^{1-\alpha}$ to reduce the Bernoulli equation to a linear equation. (4 points)
b) Calculate the general solution of

$$\dot{x} = x + t/x. \qquad \text{(4 points)}$$

Exercise 10. Let $M \subseteq \mathbb{R} \times \mathbb{R}^n$, and $f\colon M \to \mathbb{R}^m$. We say that f satsifies a *Lipschitz condition* with respect to x (on the set M), if there is $L \in [0, \infty)$ with

$$\|f(t, x_1) - f(t, x_2)\| \leq L\,\|x_1 - x_2\| \quad \text{for all } (t, x_1), (t, x_2) \in M.$$

We say that f satisfies a *local Lipschitz condition* with respect to x if each point of M has a neighborhood on which f satisfies a Lipschitz condition with respect to x (the constant L might depend on the neighborhood).

Recall that the set M is *convex* if for each $x, y \in M$ also the connecting line segment $\lambda x + (1 - \lambda)y$ $(0 \leq \lambda \leq 1)$ belongs to M.

Show for an open set $M \subseteq \mathbb{R} \times \mathbb{R}^n$:

a) If the derivative $\frac{\partial f}{\partial x}$ exists and is bounded and if M is convex then f satisfies a Lipschitz condition with respect to x on M. (2 points)
b) Conclude: If $\frac{\partial f}{\partial x}$ exists and is continuous, then f satisfies a local Lipschitz condition with respect to x on M. (2 points)
c) Can one omit the hypothesis that M is convex in a)? (2 points)
d) Is there a function f which satisfies a Lipschitz condition with respect to x on $M = \mathbb{R} \times \mathbb{R}^n$ although $\frac{\partial f}{\partial x}$ does not exist on M? (2 points)