3. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, May 13, 2016

Exercise 11. Calculate the general solution of the following differential equations:

a) $tx' = x(\ln x - \ln t)$. (2 points)

b) $2t^2x' = t^2 + x^2$. (2 points)

c) $tx' = x + \sqrt{t^2 - x^2}$. (2 points)

Exercise 12. Show that the task to calculate the solution of the scalar equation

$$x' = f\Big(\frac{at + bx + \gamma}{ct + dx + \delta}\Big)$$

can always be reduced to calculate the solution of an equation with separated variables. More precisely, proceed as follows:

a) If $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \neq 0$, use the substitution

$$y(t) = \frac{x(t - t_0) + x_0}{t},$$

(here, $x(t - t_0)$ means x evaluated at $t - t_0$, not a product), where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} t_0 \\ x_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}\right)$, and differentiate $ty(t)$ to obtain eventually a differential equation for y. (4 points)

Remark. Alternatively, you can also use the substitution $z(t) = x(t - t_0) + x_0$ to reduce the problem to a 0-homogeneous equation for z. However, as you have seen in the lecture, you will have to use a further substitution to obtain an equation with separated variables. Both approaches are actually just a different notation for the same thing, because $y(t) = z(t)/t$ corresponds exactly to the substitution from the lecture.

b) If $\det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 0$ use after some case distinction a rather obvious substitution in each case. (3 points)

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Exercise 13. Let $M \subseteq \mathbb{R}^N$, and let $f\colon M \to \mathbb{R}^N$ and $\mu\colon M \to \mathbb{R} \setminus \{0\}$ be continuous. The aim of this exercise is to show that the autonomous system

$$\dot{x} = f(x) \tag{$*$}$$

and the "scaled by the scalar factor $\mu \neq 0$" system

$$\dot{y} = \mu(y) f(y) \tag{$**$}$$

have the same trajectories: The solutions differ only up to a rescaling of time (in case $\mu(y(t)) > 0$ for all t) or rescaling and inversion of time (in case $\mu(y(t)) < 0$), respectively.
More precisely, show the following assertions:

a) If x is a solution of $(*)$ in an interval $(a,b) \ni s_0$ and φ is a solution of the autonomous 1-dimensional initial value problem

$$\varphi'(s) = \mu\big(x(\varphi(s))\big), \quad \varphi(t_0) = s_0, \tag{$***$}$$

then $\varphi\colon (a,b) \to (\alpha,\beta) \ni t_0$ is a diffeomeorphism for appropriate numbers $\alpha < \beta$, and the time-reparametrization $y := x \circ \varphi$, that is $y(t) := x\big(\varphi(t)\big)$, is a solution of $(**)$. (2 + 2 points)

b) Conversely, if y is a solution of $(**)$ in an interval $(\alpha,\beta) \ni t_0$ and

$$\psi(t) := s_0 + \int_{t_0}^{t} \mu\big(y(\tau)\big)\, d\tau,$$

then $\psi\colon (\alpha,\beta) \to (a,b) \ni s_0$ is a diffeomorphism with appropriate numbers $a < b$, and with $\varphi := \psi^{-1}$ the time reparametrization $x = y \circ \varphi$ is a solution of $(*)$. (2 points + 3 extra points)

Obviously, the solution x on (a,b) produces the same trajectory as the solution y on (α,β), and the solutions pass through the trajectory in the same (or oppositve) direction if and only if $\varphi', \psi' > 0$ (or < 0, respectively). The latter holds if and only if $\mu \circ y > 0$ (or < 0) on (α,β), respectively.

Exercise 14. Note that Exercise 13 implies in particular that the idea of Euler multipliers can also be used for Hamiltonian systems: The systems

$$\begin{aligned} \dot{x} &= G(x,y) \\ \dot{y} &= -F(x,y) \end{aligned}$$

and

$$\begin{aligned} \dot{x} &= \mu(x,y) G(x,y) \\ \dot{y} &= -\mu(x,y) F(x,y) \end{aligned}$$

have the same trajectories if $\mu(x,y) \neq 0$. Use this for the prey-predator system

$$\begin{aligned} \dot{x} &= \alpha x - \beta x y \\ \dot{y} &= \gamma x y - \delta y \end{aligned}$$

to find a (nondegerate) function H such that, when we restrict our interest to trajectories inside of the domain $M = \{(x,y) : x, y > 0\}$, all these trajectories lie in the level sets $H(x,y) \equiv c$. (5 points)
Hint. Use the Euler multiplier $\mu(x,y) = 1/(xy)$.