4. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, May 20, 2016

Aufgabe 15. Which of the following functions satisfies a local and/or global Lipschitz condition with respect to x on the specified sets?

a) $f(t,x) = t^2\sqrt{x}$ on $[0,1] \times [0,1]$. (2 points)

b) $f(t,x) = \sqrt{t}x^2$ on $[0,1] \times [0,1]$. (2 points)

c) $f(t,x) = \sqrt{t}x^2$ on $[0,1] \times \mathbb{R}$. (2 points)

d) $f(t,x) = t \sin x$ on $[0,1] \times \mathbb{R}$. (2 points)

e) $f(t,x) = t \sin x$ on $\mathbb{R} \times \mathbb{R}$. (2 points)

Exercise 16. The so-called *Riccati equation* has the form

$$x' = f(t)x + g(t)x^2 + h(t).$$

It cannot be solved explicitly, in general. However, sometimes one can guess a particular solution x_p.

a) Prove that in this case the substitution $y = x - x_p$ helps to solve the Riccati equation. (2 points)
Hint. Recall that Bernoulli's equation can be solved.

b) Solve $x' = x^2 + 2tx + 2$. (4 points)
Hint. One solution is $x(t) = -1/t$.

Let $M \subseteq \mathbb{R}^{n+1}$, and $f\colon M \to \mathbb{R}^n$. We say that f has the *local forward uniqueness property* on M if for each $(t_0, x_0) \in M$ there are $\varepsilon_1, \delta > 0$ such that for every $\varepsilon \in (0, \varepsilon_1)$ there is at most one solution of the initial value problem

$$x' = f(t,x), \quad x(t_0) = x_0 \qquad \text{(IVP)}$$

on $I = [t_0, t_0 + \varepsilon]$ which satisfies the estimate $\|x(t) - x_0\| \leq \delta$ for all $t \in I$.
The *local backward uniqueness property* is defined analogously with I being replaced by $I = [t_0 - \varepsilon, t_0]$.

Exercise 17. a) Show that if $f\colon M \to \mathbb{R}^n$ is continuous and locally Lipschitz on M with respect to the last argument, then f has the local forward and backward uniqueness property on M. (M does not need to be open.) (4 points)
Hint. Put $Y := C(I, \mathbb{R}^n)$ with the norm $\|x\| = \max_{t\in I} \|x(t)\|$,

$$X := \{x \in Y \mid (t, x(t)) \in M \text{ and } \|x(t) - x_0\| \leq \delta \text{ for all } t \in I\},$$

define $\Phi\colon X \to Y$ as in the proof of the Picard-Lindelöf theorem, and recall how the uniqueness is shown in the proof of Banach's fixed point theorem.

b) Show that if f has the local forward (or backward) uniqueness property on M, then f has the *global* forward (or backward) uniqueness property, i.e. for *every* interval J with $t_0 = \min J$ (or $t_0 = \max J$) the problem (IVP) has at most one solution on J. (4 points)
Hint. Assume that $y, z\colon J \to \mathbb{R}^n$ are two different solutions on J. Then necessarily y and z are continuous (since differentiable), and their graph lies in M. Now elaborate the argument used in the proof of Corollary 3.1.