5. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, May 27, 2016

Exercise 18. Let $M \subseteq \mathbb{R} \times \mathbb{R}^n$, and let $f\colon M \to \mathbb{R}^m$ satisfy a local Lipschitz condition with respect to x. Show the following assertions.

a) If $t \mapsto f(t, x)$ is continuous for every x, then f is continuous on M. (2 points)
Remark. Recall that the continuity of $t \mapsto f(t, x)$ and of $x \mapsto f(t, x)$ does in general not imply the continuity of f. So you have to prove that this is different under a local Lipschit condition with respect to x.

b) f does in general not satisfy a (global) Lipschitz condition with respect to x on M, even if M is bounded and convex. (2 points)

c) If M is compact and convex then f satisfies a (global) Lipschitz condition with respect to x on M. (4 points)
Can the convexity of M be dropped in this assertion? (3 extra points)

Exercise 19. Let $g\colon [a, b] \to (0, \infty)$ be continuous. Equip $X = C([a, b], \mathbb{R}^n)$ with the two different norms

$$\|x\|_\infty := \max_{t\in[a,b]} \|x(t)\| \quad \text{and} \quad \|x\|_g := \max_{t\in[a,b]} \|g(t)x(t)\| .$$

(The former is the norm from the lecture.)
Show that the norms are equivalent, that is, there are constants $c_1, c_2 \in (0, \infty)$ such that

$$c_1 \|x\|_\infty \le \|x\|_g \le c_2 \|x\|_\infty ,$$

and conclude that $(X, \|\cdot\|_g)$ is a Banach space. (2 + 2 points)

Exercise 20. Let $M = [t_0, b] \times \mathbb{R}^n$, and let $f\colon M \to \mathbb{R}^n$ satisfy a Lipschitz condition with respect to x on M with a constant $L \in [0, \infty)$. Let $x_0 \in \mathbb{R}^n$. Consider for some $\gamma > L$ the Banach space $(X, \|\cdot\|_g)$ from Exercise 19 with $g(t) = e^{-\gamma t}$ and $[a, b] = [t_0, b]$.

a) Show that $\Phi\colon X \to X$, given by

$$\Phi(x)(t) := x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

is a contraction of X with contraction constant at most $q = L/\gamma$. (4 points)
Remark. With the function $g(t) = e^{\gamma t}$, one could obtain a similar result with $[a, b] = [a, t_0]$.

b) Conclude that the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0$$

has a unique solution in $[t_0, b]$.
Compare the latter result with the local existence and uniqueness thoerem of Picard-Lindelöf, i.e. describe in one or two brief sentences what is the essential difference in the conclusion and in the assumption. (2 + 2 points)