6. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, June 3, 2016

Exercise 21. Let X be a complete metric space, and suppose $\Phi\colon X \to X$ satisfies

a) $d(\Phi(x), \Phi(y)) \le d(x, y)$ for all $x, y \in X$.
b) $d(\Phi(x), \Phi(y)) < d(x, y)$ for all $x, y \in X$, $x \ne y$.

Must Φ have a fixed point? Can there be several fixed points? (2 + 2 + 2 points)

Exercise 22. Let $\Phi\colon X \to X$ be a contraction with constant $q < 1$ and fixed point x_*. Given $x_0 \in X$, let $x_n := \Phi^n(x_0)$. Show the error estimate

$$d(x_{m+k}, x_*) \le \frac{q^k}{1-q} d(x_{m+1}, x_m) \quad \text{for all } m, k = 0, 1, \dots \qquad \text{(2 points)}$$

Remark. The special case $m = 0$, $k = n$ is the error estimate from the lecture. The special case $k = 1$

$$d(x_{m+1}, x_*) \le \frac{q}{1-q} d(x_{m+1}, x_m)$$

means that when one calculates the approximations x_m iteratively, one can also track a corresponding error estimate. We employ this in Exercise 23.

Exercise 23. The aim of this exercise is to find an approximate solution to the particular Riccati equation

$$x' = t^2 - x^2, \quad x(0) = 0,$$

such that the approximate solution differs for every $t \in [0, 1/2]$ from the exact solution at most by 10^{-3}.

Therefore, we consider for $f(t, x) = t^2 - x^2$ the operator

$$\Phi(x)(t) = 0 + \int_0^t f(t, x(t))\, dt$$

from the proof of the Picard-Lindelöf theorem in the space

$$X = \{x \in C([0, 1/2]) : |x(t) - 0| \le b\}$$

with the metric $d(x, y) = \|x - y\|$, where

$$\|x\| = \max_{t \in [0,1/2]} |x(t)|\,.$$

a) Find some (for later purposes hopefully as small as possible) $b > 0$, such that $\Phi\colon X \to X$. (2 Punkte)
b) Starting from $x_0 \in X$, the Picard iterations are defined as $x_n := \Phi^n(x_0)$. Starting from $x_0(t) \equiv 0$, calculate x_1 and x_2. (2 Punkte)
c) What is $d(x_2, x_1)$? (1 Punkt)
d) Determine some (for later purposes hopefully very small) Lipschitz constant of f with respect to x on the cuboid $[0, 1/2] \times [-b, b]$ and use it to obtain a (small) contraction constant q for Φ. (2 + 2 Punkte)
e) Use the previous infomation to show with the aid of Exercise 22 that x_2 has the required approximation property. (2 Punkte)

   Remark. Depending on your previous estimates for q your error estimate will probably be *much* better than 10^{-3}.

Exercise 24. For the following sets $M \subseteq X = C([0, 1])$, is $\overline{M}$ equicontinuous and/or compact?

a) $M = \{x \in X : x(0) = 0, x(1) = 1, 0 \le x(t) \le 1 \text{ for all } t \in [0, 1]\}$. (2 points)
b) $M = \{x \in X : x(0) = 0, x \text{ is Lipschitz with constant } L\}$ (for a fixed given L). (2 points)
c) $M = \{x_1, x_2, \dots\}$ such that $x_n(t) = t^n$. (2 points)