7. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, June 10, 2016

Exercise 25. Let X be an infinite-dimensional Banch space. It can be shown that there is an (artificially constructed) continuous function $f\colon \mathbb{R} \times X \to X$ such that there is no $\varepsilon > 0$ such that the initial value problem

$$x' = f(t, x), \quad x(0) = 0$$

has a solution on $(-\varepsilon, \varepsilon)$. (In many Banach spaces, one can even construct an autonomous such f i.e. independent of t.)

Answer (very sketchy with just a few words) the following questions:

a) Why does the above assertion not contradict Peano's theorem? Which step of the proof from the lecture (using Euler polygones) breaks down? (4 points)

Hint. Use that the closed unit ball $K(X) = \{x \in X : \|x\| \leq 1\}$ of X fails to be compact.

b) Might the above constructed f be locally Lipschitz with respect to x? (3 points)

Hint. Use without proof that the existence of an integral is not a problem, that is: There is an integral (called *Bochner integral*) for functions $f\colon [a, b] \to X$ which has (practically) all "analogous" properties of the Lebesgue integral, including the estimate

$$\left\| \int_a^b f(t)\, dt \right\| \leq \int_a^b \|f(t)\|\, dt$$

and the property that if f is Bochner integrable (or at least continuous) and $t_0 \in [a, b]$ then the integral function $F(t) = \int_{t_0}^t f(s)\, ds$ exists on $[a, b]$ and satisfies $F'(t) = f(t)$ for almost all (or all, respectively) $t \in [a, b]$.

Exercise 26. Let $M \subseteq \mathbb{R}^n$ and $L > 0$. For $\varepsilon \geq 0$ let $f, f_\varepsilon \in C(M, \mathbb{R}^n)$ be such that

$$\|f(t, x) - f_\varepsilon(t, y)\| \leq L\,\|x - y\| + \varepsilon \quad \text{for all } (t, x), (t, y) \in M,$$

and let $(t_0, u_0), (t_0, u_\varepsilon) \in M$ satisfy $\|u_0 - u_\varepsilon\| \leq \varepsilon$. Use Gronwall's lemma to calculate an error function $E_\varepsilon\colon [a, b] \to [0, \infty)$ such that if x and x_ε are respective solutions of the unperturbed or perturbed initial value problems

$$\begin{Bmatrix} x' = f(t, x) \\ x(t_0) = u_0 \end{Bmatrix} \qquad \begin{Bmatrix} x_\varepsilon' = f_\varepsilon(t, x_\varepsilon) \\ x_\varepsilon(t_0) = u_\varepsilon \end{Bmatrix}$$

on $[a, b] \ni t_0$, then $\|x(t) - x_\varepsilon(t)\| \leq E_\varepsilon(t)$ for all $t \in [a, b]$. Moreover, $E_\varepsilon \to E_0 = 0$ unifomly on $[a, b]$. (4 points)

Argue why it follows, as a special case, that if f satisfies a Lipschitz condition on M then the first initial problem can have at most one solution on $[a, b]$ (i.e., as a side result you have actually given an alternative proof of the uniqueness assertion of the Picard-Lindelöf theorem). (2 points)

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Let $M \subseteq \mathbb{R} \times \mathbb{R}^n$ and $f\colon M \to \mathbb{R}^n$. We say that f satisfies on M a one-sided Lipschitz condition with respect to x with the constant L if

$$\langle f(t,x) - f(t,y), x - y \rangle \leq L \, \|x - y\|^2 \quad \text{for all } (t,x), (t,y) \in M.$$

Exercise 27. a) Show: If f and g satisfy a one-sided Lipschitz condition with respect to x with some constant on M, then also the function $\alpha f + \beta g$ does for every $\alpha, \beta \in [0, \infty)$. (2 points)

b) Do the functions satisfying a one-sided Lipschitz condition even form a vector space, that is, is the previous assertion even true for every $\alpha, \beta \in \mathbb{R}$? (2 points)

c) Which of the following functions satisfy a one-sided Lipschitz condition with a constant with respect to x? Which of these satisfies a Lipschitz condition (not one-sided) with respect to x?

(i) $f(t,x) = \operatorname{sgn} x$ on $\mathbb{R}^2$ (2 points)

(ii) $f(t,x) = -\operatorname{sgn} x$ on $\mathbb{R}^2$ (2 points)

(iii) $f(t,x) = \begin{cases} x+1 & \text{if } x < 0, \\ -\sqrt{x} & \text{if } x \geq 0, \end{cases}$ on $\mathbb{R}^2$ (2 points)

(iv) $f(t,x_1,x_2) := \begin{pmatrix} x_2 - x_1^3 \\ x_1 - x_2^3 \end{pmatrix}$ on $\mathbb{R}^3$ (4 points)