8. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, June 17, 2016

Exercise 28. Let $a, b \in C([t_0, T], \mathbb{R})$, $B(t) = \int_{t_0}^{t} b(s)\,ds$, and $x_0 \in \mathbb{R}$.

a) Show by using the technique of exact equations and the Euler multiplier $e^{-B(t)}$ that x is a solution of the inhomogeneous linear initial value problem

$$x'(t) = b(t)x(t) + a(t) \quad \text{for all } t \in [t_0, T], \quad x(t_0) = x_0$$

if and only if x satisfies the variation-of-constants formula

$$x(t) = e^{B(t)}x_0 + \int_{t_0}^{t} e^{B(t)-B(s)}a(s)\,ds \quad \text{for all } t \in [t_0.T].$$

(The formula was of course obtained in the lecture, but by a completely different method.) (4 points)

b) Use the above proof to show the following differential form of the Gronwall lemma: If $x \in C([t_0, T]) \cap C^1((t_0, T))$ satisfies the implicit inequalities

$$x'(t) \le b(t)x(t) + a(t) \quad \text{for all } t \in (t_0, T), \quad x(t_0) \le x_0,$$

then x also satisfies the explicit inequalities

$$x(t) \le e^{B(t)}x_0 + \int_{t_0}^{t} e^{B(t)-B(s)}a(s)\,ds \quad \text{for all } t \in [t_0, T].$$

(The formulas are almost the same as in the first part, only with "$\le$"; in particular, the explicit inquality is the "best possible".) (4 points)

Remark. In contrast to the lecture, this variant of the Gronwall lemma does not require that a, b, or x_0 are nonnegative. We actually have to use this form of the Gronwall lemma with $a \equiv 0$ and $b = 2\ell$ in the proof of Theorem 5.5 in the case that ℓ assumes also negative values.

Hint. Let H be the Hamilton/potential function from the first part, and let $x \in C^1([t_0, T])$ satisfy the implicit inequality. Show the monotonicity of $h(t) = H(t, x(t))$ by calculating its derivative, and use this to obtain the sign of $h(t) - h(t_0)$. Write down the latter as an inequality.

Exercise 29. Let I be an interval, $A \in C(I, \mathbb{R}^{n\times n})$, and let X be a fundamental matrix for $x' = A(t)x$. Show:

a) For each regular matrix C also $Y(t) = X(t)C$ is a fundamental matrix. (2 points)

b) One obtains each fundamental matrix in this way. (2 points)

c) Für which matrices C is $Y(t) = CX(t)$ a fundamental matrix? (4 points)

Exercise 30. a) Calculate on $I = (0, \infty)$ a fundamental system for

$$x_1' = \frac{1}{t}x_1 - x_2,$$
$$x_2' = \frac{1}{t^2}x_1 + \frac{2}{t}x_2. \qquad \text{(4 points)}$$

Hint. One solution of this problem is $x_1(t) = t^2$, $x_2(t) = -t$.

b) Find one particular solution of the following problem on $I = (0, \infty)$:

$$x_1' = \frac{1}{t}x_1 - x_2 + t^3,$$
$$x_2' = \frac{1}{t^2}x_1 + \frac{2}{t}x_2 + t. \qquad \text{(4 points)}$$