9. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, June 24, 2016

Exercise 31. Calculate the fundamental matrix X satisfying $X(0) = E$ for the linear homogeneous problem

$$x' = 2ty,$$
$$y' = -2tx,$$

by Picard's iteration, starting from $X_0(t) \equiv E$. (6 points)

Exercise 32. Recall that, by definition, $e^{tA} := X_A(t)$, where X_A is that (unique) fundamental matrix of $x' = Ax$ satisfying $X_A(0) = E$.
From this definition it is not clear that e^{tA} is well-defined, that is, that it really depends only on the matrix tA instead of the couple (t, A). Show, only using the definition, that $e^{s(tA)} = e^{(st)A}$ for every $s \in \mathbb{R}$ (which for $s = 1$ implies that e^{tA} is well-defined). (2 points)

Exercise 33. Let I be an interval, $t_0 \in I$, and $A \in C(I, \mathbb{R}^{n\times n})$. Show:

a) If A has the commutativity property

$$A(t)A(s) = A(s)A(t) \quad \text{for all } t, s \in I,$$

then A also has the commutativity property

$$A(t)\int_{t_0}^{t} A(s)\,ds = \int_{t_0}^{t} A(s)\,ds\,A(t) \quad \text{for all } t \in I. \qquad \text{(1 point)}$$

b) If A satisfies at least one of the above commutativity properties, then the fundamental matrix X of $x' = A(t)x$ satisfying $X(t_0) = E$ is given by

$$X(t) = \exp\Big(\int_{t_0}^{t} A(s)\,ds\Big) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\int_{t_0}^{t} A(s)\,ds\Big)^n.$$

Be very precise in your arguments! (E.g., where do you actually use the commutativity? Why can you exchange differentiation and series?) (5 points)

Remark. In particular, if A has one of the above commutativity properties and $b \in C(I, \mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$, then the solution of the initial value problem $x' = A(t)x + b(t)$, $x(t_0) = x_0$, is given by the variation-of-constants formula

$$x(t) = e^{\int_{t_0}^{t} A(s)\,ds} x_0 + \int_{t_0}^{t} e^{\int_{s}^{t} A(\sigma)\,d\sigma} b(s)\,ds.$$

Warning: Without the commutativity property, the above formulas are false, in general! The commutativity property is "almost" necessary for them!

Examples for solving equations with constant coefficients will appear on the next exercise sheet. Instead, here is a small "independent" exercise with an important idea which we will generalize soon.

Exercise 34. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, and let $x\colon \mathbb{R} \to \mathbb{R}$ be a solution of the differential equation $x' = f(x)$ with $x(0) > 0$.

a) Show that there is some $t_0 > 0$ such that $x(t) > 0$ for all $t \in [0, t_0)$. (1 point)

b) Show that in case $f(0) > 0$ even $x(t) > 0$ for all $t \geq 0$. (3 points)

c) Does it follow even in case $f(0) \geq 0$ that $x(t) > 0$ or at least $x(t) \geq 0$ for all $t \geq 0$? (3 + 1 extra points)

d) Show that if f has the forward uniqueness property then $x(0) \geq 0$ and $f(0) \geq 0$ imply $x(t) \geq 0$ for all $t \geq 0$. (4 extra points)