10. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, July 1, 2016

Exercise 35. Find a *real* fundamental matrix for $x' = Ax$ with

a)

$$A := \begin{pmatrix} 2 & 4 \\ -4 & 1 \end{pmatrix}. \qquad \text{(5 points)}$$

b)

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & 4 & 0 & 2 & 1 \\ 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ -1 & 1 & 0 & 1 & 3 \end{pmatrix}. \qquad \text{(5 points)}$$

Hint. The only eigenvalues are 2 and 3.

Exercise 36. Calculate the general solution of

a) $x'' + 2x' + x = t^3 + e^{-t}$. (6 points)

b) $x''' + 2x'' = t^3$. (4 extra points)

Exercise 37. The D'Alembert reduction from the lecture is usually not appropriate for

$$a_0(t)x + a_1(t)x' + \cdots + a_n(t)x^{(n)} = 0,$$

because the "reduced" system cannot be rewritten in the above form, in general. However, if a solution y with $y(t) \neq 0$ on some interval I is known, one can make on I the ansatz

$$x(t) = y(t)z(t).$$

a) Use the Leibniz rule

$$x^{(k)} = \sum_{j=0}^{k} \binom{k}{j} y^{(k-j)} z^{(j)} = y^{(k)} z + \sum_{j=1}^{k} \binom{k}{j} y^{(k-j)} z^{(j)}$$

to determine coefficients b_k such that if $(w_1, \ldots, w_{n-1})$ is a fundamental system for

$$b_1(t)w + b_2(t)w' + \cdots + b_n(t)w^{(n-1)} = 0 \qquad (*)$$

and $z_k' = w_k$ $(k = 1, \ldots, n-1)$ then $(x_1, \ldots, x_n) = (y, yz_1, \cdots, yz_{n-1})$ is a fundamental system for the original equation on I. (4 extra points)

Hint. What does $(*)$ mean in terms of z when $z' = w$? For the proof of the linear independence assume by contradiction

$$\lambda_1 x_1 + \cdots + \lambda_n x_n = 0.$$

b) Find on $I = (0, \infty)$ the general solution of

$$3x + tx' - t^2 x'' = t^3. \qquad \text{(3 + 3 extra points)}$$

Hint. One solution of the homogeneous problem is $y(t) = 1/t$.