11. Exercise for Differential Equations I (SS 2016 Väth)

Please finish until: Friday, July 8, 2016

Exercise 38. Suppose that $M \subseteq [a,b] \times \mathbb{R}^n$ is open in $[a,b] \times \mathbb{R}^n$. Let $f\colon M \to \mathbb{R}^n$ be continuous and locally Lipschitz with respect to x. Let S denote the set of all solutions on $[a,b]$ of the equation $x' = f(t,x)$. Let $S_t := \{x(t) : x \in S\}$. Show that S_t is open for every $t \in [a,b]$ and that the translation-in-time map $T\colon S_a \to S_b$ which for every $x \in S$ sends $x(a)$ to $x(b)$ is a homeomorphism, that is, continuous and bijective with a continuous inverse. (4 extra points)

Exercise 39. Show that for a 2-dimensional system $x' = Ax$ with a real matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

the equilibrium $(0,0)$ is asymptotically stable if

$$a_{11}a_{22} - a_{12}a_{21} = \det A > 0, \quad a_{11} + a_{22} = \operatorname{Trace} A < 0,$$

and unstable if $\det A < 0$ or $\operatorname{Trace} A > 0$. (4 points)

Exercise 40. Romeo and Julia love each other, but not always at the same level. Let $u(t)$ and $v(t)$ describe how much the love of Julia and Romeo deviates from the "average" level $u = 0$, $v = 0$ at time t. Assume that Julia has such a nature that her own strong love and also Romeo's strong love will cause her own love to increase, while Romeo has the opposite nature, that is, consider $x' = Ax$ with

$$x(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with $a_{11}, a_{12} > 0$ and $a_{21}, a_{22} < 0$. Assume also that

$$a_{11}a_{22} - a_{12}a_{21} = \det A > 0, \quad a_{11} + a_{22} = \operatorname{Trace} A < 0.$$

Distinguishing the two cases that the corresponding eigenvalue λ_1, λ_2 are both real or both (conjugate) complex, sketch qualitatively the phase portrait in the (u,v)-plane. (4 extra points)

Remark. All cases can actually occur. Neither the real nor the complex case are "exceptional" (although the domain of matrices in which the real case occurs has a smaller measure).

Hint. In which direction do the trajectories point in the first and third quadrant and on the u-axis? How do the solutions behave as $t \to \infty$?

Exercise 41. Given constants $\alpha, \beta, \gamma, \delta, a, b > 0$ with $\alpha\delta > a\gamma$ consider the modified prey-predator system

$$\dot{x} = \alpha x - \beta xy - ax^2,$$
$$\dot{y} = \delta xy - \gamma y - by^2.$$

(The terms with a and b mean an artificial growth limitation for x and y).

a) Show that all solutions starting in the closed first quadrant will remain in this quadrant for all time. (4 points)

   Hint. Use the uniqueness of solutions.

b) Calculate all equilibria in the first quadrant and determine their stability. (It is not necessary to discuss the cases which cannot be obtained by linearization.) (4 extra points)