Characterization of 1-dimensional Manifolds

Martin Väth

Theorem. *Every separable metrizable* 1*-dimensional connected* C^n *manifold (with or without boundary) is* C^n*-diffeomorphic to either the circle* $S^1 := S_1(0)$ *in* $\mathbb{R}^2$ *or to an interval.*

The separability is required to exclude pathological manifolds like the "long line" (uncountably many intervals of length 1 stuck together). Note that a one-dimensional *manifold with boundary* is defined analogouosly to a manifold without boundary, only that for the chart(s) onto the boundary point, the preimage of the manifold only has to be an half-open interval (not an open interval).

A proof can be found in [1], although some parts of that proof are not given in much a detail. For compact manifolds the proof can be slightly simplified (though the main ideas are the same).

Here is a more verbose proof of this special case by myself (based on the proof from [1]) which appeared as [2, Theorem 9.15].

Theorem. *Every compact connected* 1*-dimensional* C^1 *manifold* X *(with or without boundary) is diffeomorphic to either the circle* $S^1 := S_1(0)$ *in* $\mathbb{R}^2$ *or to* $[0, 1]$.

Proof. By shrinking the charts if necessary, we can assume that X is covered by open sets, each of which is diffeomorphic to an interval. By the compactness, we have $X = U_1 \cup \cdots \cup U_n$ such that each U_k is diffeomorphic to an interval. Now if X is not diffeomorphic to S^1, we show inductively that there are pairwise different $k_1, \dots, k_j$ and diffeomorphisms f_j of an interval I_j onto $X_j := U_{k_1} \cup \cdots \cup U_{k_j}$. Then we are done for $j = n$.

For the induction start, we put $k_1 := 1$. For the induction step, note that the connectedness of X implies that there is some k_{j+1} different from $k_1, \dots, k_j$ such that $U := U_{k_{j+1}}$ intersects X_j. Let f be a diffeomorphism of an interval I onto U. We put now $M := f_j^{-1}(U \cap U_j)$ and consider

$$\Gamma := \operatorname{graph} f^{-1} \circ f_j|_M = \{(t, s) \in I_j \times I : f(s) = f_j(t)\}$$

as a subset of $I_j \times I$. Note that $g = f^{-1} \circ f_j|_M$ is a diffeomorphism (of M onto $f^{-1}(U \cap U_j)$), in particular Γ is the graph of the one-to-one C^1 function g with nonzero derivative. Moreover, if $(t, s) \in \Gamma$ belongs to the interior of $I_j \times I$ then Γ extends to the left and right of t, that is, Γ cannot end in the interior of the rectangle $I_j \times I$. Since, by the injectivity of g, Γ intersects each side of the rectangle at most once, it follows that there are at most two components of Γ, each starting and ending at different sides of the rectangle. If there is only one component, M is an interval. In this case, it is clear that we can (after reparametrizing f) "concatenate" f_j and f to a homeomorphism of $X_j \cup U$ to an interval. Thus, assume Γ has two components, each connecting two sides of the rectangle. By the injectivity of g, these must be "neighboring" sides, M is the union of two disjoint open in M intervals M_1 and M_2, and by the injectivity of g the intervals $g(M_1)$ and $g(M_2)$ cannot intersect. Filling the "gaps" between the intervals M_1 and M_2 using f_j and between $g(M_1)$ and $g(M_2)$ using f, we see that there is a diffeomorphism h of S^1 onto $X_j \cup U$. Since $h(S^1) = X_j \cup U$ is open and compact and thus closed in X, the connectedness of X implies $X = h(S^1)$. Hence, X is diffeomorphic to S^1 which we had excluded for the induction. □

References

[1] Milnor, J. W., *Topology from the differential viewpoint*, University Press of Virginia, Charlottesville, 1965.

[2] Väth, M., *Topological analysis. From the basics to the triple degree for nonlinear Fredholm inclusions*, de Gruyter, Berlin, New York, 2012.