2. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, November 2, 2016

Exercise 4. Let A be a complex $(n \times n)$-matrix. We denote the Euclidean norm of $\mathbb{C}^n$ by $\|\cdot\|_2$. The *spectral radius* of A is the number $r(A) := \max\{|\lambda| : \lambda \in \sigma(A)\}$.

a) Show that for every complex $\varepsilon \neq 0$ there is a regular transformation matrix T such that TAT^{-1} is in "ε-Jordan normal form" where the ε indicates that "1" is replaced by "ε" in the "usual" Jordan normal form. (2 Points)

Hint. Consider $T_0 = \operatorname{diag}(\varepsilon, \varepsilon^2, \dots, \varepsilon^n)$.

b) Prove that $r(A)$ is (up to small ε) a universal operator norm of A; more precisely show the following asssertions:

1. If $\|\cdot\|$ is some norm on $\mathbb{C}^n$ then the corresponding operator norm must satisfy $\|A\| \geq r(A)$. (1 Point)
2. Conversely, for every $\varepsilon > 0$ there is a transformation T such if $\mathbb{C}^n$ is equipped with the norm $\|x\| = \|Tx\|_2$, then the corresponding operator norm of A satisfies $\|A\|_T \leq r(A) + \varepsilon$. (2 Points)

c) Conclude the following generalization of the lemma from the lecture. If $\|\cdot\|$ denotes any matrix norm then:

1. For every $\varepsilon > 0$ there is $C > 0$ such that $\|A^k\| \leq C(r(A) + \varepsilon)^k$ for $k = 0, 1, \dots$ (1 Point)
2. If every eigenvalue $\lambda \in \sigma(A)$ satisfies $|\lambda| \geq c > 0$ then for every $0 < \varepsilon < c$ there is $C > 0$ such that $\|A^{-k}\| \leq C(c - \varepsilon)^{-k}$ for $k = 0, 1, \dots$ (2 Points)

d) Show that if every $\lambda \in \sigma(A)$ satisfies $\operatorname{Re}\lambda < \alpha$, then there is a transformation T such that $\|e^{tA}\|_T \leq e^{\alpha t}$ for all $t \geq 0$, where the left-hand side denotes the operator norm introduced earlier. (4 Points)

e) Conclude in case $\alpha < 0$ that $t \mapsto \|e^{tA}u\|_T$ is strictly decreasing on $\mathbb{R}$ for every $u \neq 0$. (3 Points)

Remark. Analogous assertions can be shown for real matrices (with real ε and T) and when $\mathbb{C}^n$ is replaced by $\mathbb{R}^n$: Esssentially one has to replace the complex Jordan normal form by the so-called real Jordan normal form.

Exercise 5. Let A and B be two hyperbolic $(n \times n)$-matrices, both with $k \geq 0$ eigenvalues of positive real part and $\ell \geq 0$ eigenvalues of negative real part (counted with algebraic multiplicity). Show that the flows corresponding to $x' = Ax$ and $y' = By$ are globally C^0-conjugate. (12 Points)

Hint. Show first that it suffices to consider the cases $k = 0$ and $\ell = 0$. Show then that is suffices to consider the case $k = 0$. Show finally that it suffices to consider the case $B = -E$. To establish the assertion in the latter case, consider

$$\varphi(u) := \begin{cases} e^{-\ln\|u\| A} \frac{u}{\|u\|} & \text{if } u \neq 0, \\ 0 & \text{if } u = 0, \end{cases}$$

where $\|\cdot\|$ is some norm such that $t \mapsto \|e^{tA}u\|$ is strictly decreasing on $\mathbb{R}$ for every $u \neq 0$ (cf. Exercise 4). To simplify the calculation, use without proof the famous topological open mapping theorem: Every continuous bijection φ of $\mathbb{R}^n$ is automatically a homeomorphism (i.e., its inverse is automatically continuous or – equivalently – φ maps open sets onto open sets).