3. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, November 9, 2016

Exercise 6. Show by means of an example that the coordinate transform φ in the (discrete) Hartman-Grobman theorem is far from being unique without the requirement that $id - \varphi$ be bounded. More precisely, give an example of two continuous flows Φ and Ψ on $\mathbb{R}^2$ for which there is even a whole 2-parameter family of homeomorphisms $\varphi_{a,b} \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\varphi_{a,b} \circ \Phi_t = \Psi_t \circ \varphi_{a,b}$$

holds even globally for all t and for all (a,b) from a nonempty open subset of $\mathbb{R}^2$ (and such that $(a,b) \mapsto \varphi_{a,b}$ is one-to-one). (5 Points)

A set W is called *forward/backward invariant* (or *positively/negatively invariant*) under a continous (or discrete) semiflow/flow Φ if $\Phi(t, W)$ is defined and contained in W for every $t > 0$ or $t < 0$, respectively.

Exercise 7. Let $M \subseteq \mathbb{R}^n$ be open and $f \in C^1(M, \mathbb{R}^n)$. Let V be a Ljapunov function for $x' = f(x)$ on M.

a) Show that for every $c > 0$ the sublevel sets $V^{-1}((-\infty, c])$ and $V^{-1}((-\infty, c))$ are "forward invariant", and that the superlevel sets $V^{-1}([c, \infty))$ and $V^{-1}((c, \infty))$ are "backward invariant", if we relax the corresponding definition by requiring only $\Phi(t, u) \in W$ for those $t > 0$ or $t < 0$ and $u \in W$ for which $\Phi(t, u)$ is defined. (3 Points)

b) Show that if the set $V^{-1}((-\infty, c])$ or $V^{-1}([c, \infty))$ is compact, then this set is forward or backward invariant, respectively, even without the above relaxation of the definition. (3 Points)

c) Can the hypothesis about compactness be dropped? (2 Points)

Exercise 8. Show that the simplifying artificial hypothesis $M = \{x_0\}$ in our main theorem about the Ljapunov function can actually be dropped. More precisely, show the following.

Let $U \subseteq \mathbb{R}^n$ be open, $f \in C^1(U, \mathbb{R}^n)$, Φ be the corresponding flow, and $V \in C^1(U, \mathbb{R})$ be a Ljapunov function on U. Let $M \subseteq U$ be compact, $m := \max_{u \in M} V(u)$, and let $K \subseteq U$ be a compact neighborhood of M such that $V(u) > m$ and $\dot{V}(u) < 0$ for all $u \in K \setminus M$.

Show that M is asymptotically stable. (6 Points)

Hint. We have already shown that M is stable. (Use this without repeating the proof.) If $\overline{u}$ is such that $\Phi(t, \overline{u}) \in K$ for all $t > 0$, distinguish the case that there is some $t \geq 0$ with $\Phi(t, \overline{u}) \in M$ or not. In one of the cases one can proceed as in the lecture.