4. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, November 16, 2016

In the following exercise, we use the following (weak) definition of invariance: We call $M \subseteq U$ *forward/backward invariant* (or *positively/negatively invariant*) if $\Phi(t,u) \in M$ for all $u \in M$ and all $t > 0$ or $t < 0$, respectively, for which $\Phi(t,u)$ is defined.
(Actually, this definition is more usual in literature.)

Exercise 9. Let $U \subseteq \mathbb{R}^n$ be open, $f \in C^1(U, \mathbb{R}^n)$, and Φ be the flow associated to $x' = f(x)$.

a) Show that if $M \subseteq U$ is closed in $\mathbb{R}^n$ then M is forward invariant if and only if for every $u \in M$ and every $\varepsilon > 0$ there is $t \in (0, \varepsilon)$ with $\Phi(t,u) \in M$. (3 Points)

b) Assume that for every $u \in M$ there is $\varepsilon > 0$ with $\Phi([0,\varepsilon), u) \subseteq M$, but M is not necessarily closed. Does it follow that M is forward invariant? (2 Points)

c) Show that for every $u \in U$ there holds the linearization

$$\Phi(t,u) = u + tf(u) + o(t), \text{ where } o(t)/t \to 0 \text{ as } t \to 0. \qquad (\circ)$$

Conclude that if M is forward invariant then there holds the subtangency condition

$$\frac{\operatorname{dist}(u + tf(u), M)}{t} \to 0 \quad \text{as } t \to 0^+ \text{ for every } u \in M. \qquad (*)$$

(4 Points)

d) Show that if $M \subseteq U$ is nonempty and closed in $\mathbb{R}^n$ with $(*)$ then M is forward invariant. (8 extra points)

Remark. A more careful analysis shows that the assumption $(*)$ in this last assertion can even be relaxed to

$$\liminf_{t \to 0^+} \frac{\operatorname{dist}(u + tf(u), M)}{t} = 0 \quad \text{for every } u \in M.$$

Hint. For fixed $u \in M$, put $y(t) := \operatorname{dist}(\Phi(t,u), M)$, and recall that f is Lipschitz continuous in a neighborhood of u with some constant L. Show that for every small $t \geq 0$ there holds

$$y'(t) \leq Ly(t),$$

where $y'(t)$ denotes the upper Dini right derivative $y'(t) = \limsup_{s \to 0^+} \frac{y(t+s) - y(t)}{s}$. Then use the theorem about subsolutions, applying without proof that this theorem holds actually also for the upper Dini right derivative.

To see the above inequality for fixed $t \geq 0$, justify first that there is $v \in M$ with $y(t) = \|\Phi(t,u) - v\|$. Now justify

$$\Phi(t+s, u) = \Phi(t,u) + sf(\Phi(t,u)) + o(s)$$

and

$$\|f(\Phi(t,u)) - f(v)\| \leq Ly(t)$$

and apply $(*)$ with (t,u) replaced by (s,v) to show that

$$y(t+s) \leq y(t) + sL\,\|y(t)\| + \|o(s)\|\,.$$

(continued on next page)

Exercise 10. We call a set M locally forward/backward invariant under a flow Φ if for each $u \in M$ there is $\varepsilon > 0$ with $\Phi(t, u) \in M$ for all $t \in (0, \varepsilon)$ or $t \in (-\varepsilon, 0)$, respectively.
Let $D \subseteq \mathbb{R}^{m+n}$ be open, $f \in C^1(D, \mathbb{R}^m)$, $g \in C^1(D, \mathbb{R}^n)$, and consider the flow Φ associated to the system

$$\begin{aligned} x' &= f(x, y), \\ y' &= g(x, y). \end{aligned}$$

For an open set $U \subseteq \mathbb{R}^m$ and $h \in C^1(U, \mathbb{R}^n)$ suppose that the manifold $M = \{(u, h(u)) : u \in U\}$ is contained in D. Show that M is locally forward and backward invariant if and only if the so-called tangency condition

$$g(u, h(u)) = h'(u) f(u, h(u)) \quad \text{for all } u \in U$$

is satisfied. (3 + 5 Points)

Hint. To see the sufficiency of the condition, verify the subtangency condition (∗) and use without a proof that for M the subtangency condition implies the local forward invariance of M. (The omitted proof can be shown by applying the argument of the previous exercise to intersections of M with small closed balls, using that these intersections are closed if the balls are sufficiently small; the details are clumsy.)