5. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, November 23, 2016

Exercise 11. Consider the system

$$x' = -x$$
$$y' = 2y - \lambda x^3.$$

a) Show that near 0 the local stable manifold $M = W^{s,0}(0)$ has the form
$$M = \{(u, h(u)) : u \in (-\varepsilon, \varepsilon)\}$$
with some $h \in C^\infty$, satisfying $h(0) = 0$, $h'(0) = 0$. (3 Points)

b) Make the polynomial ansatz $h(u) = a + bu + cu^2 + du^3 + o(u^3)$ with $o(u^3)/u^3 \to 0$ as $u \to 0$ and use the tangency condition to calculate a, b, c, d. (4 Points)

Exercise 12. Consider the equation $\ddot{x} = x - x^3$, that is, the 2-dimensional system

$$x' = y,$$
$$y' = x - x^3.$$

a) Show that the system is Hamiltonian with a Hamilton function $H \in C^2(\mathbb{R}^2, \mathbb{R})$ which is coercive in the sense that $H(x, y) \to \infty$ as $\|(x, y)\| \to \infty$. (3 Points)

b) Conclude that all (maximally extended) solutions are eternal, that is, they exist for all times $t \in \mathbb{R}$. (4 Points)

   Hint. Apply Exercise 7 or repeat that argument.

c) Show that there are exactly 3 stationary points z_0, $z_{\pm 1}$ with $z_{\pm 1}$ being strict global minima of H. (2 Points)

d) Show that all non-stationary solutions are either periodic or homoclinic to the stationary point z_0. (6 Points)

   Hint. Recall the following fact which you should have learnt in analysis as a consequence of the implicit function theorem: For any function $F \in C^1(U, \mathbb{R}^m)$ with open $U \subseteq \mathbb{R}^{n+m}$ and any $a \in \mathbb{R}^m$ with the property that $F'(x)$ has the maximal rank m at every point x of $M := F^{-1}(\{a\})$, the set M is an m-dimensional submanifold of $R^{n \times m}$.
   Hence, how can $M = H^{-1}(\{a\}) \setminus \{z_0\}$ look topologically?

e) Show that there are indeed (non-stationary) homoclinics. (4 Points)