6. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, November 30, 2016

When we identify in the interval $[0,1]$ the points 0 and 1, we obtain the circle S^1. The homeomorpism between these objects can be described by $\varphi \mapsto (\cos(2\pi\varphi), \sin(2\pi\varphi))$. Another description of S^1 is the identification $\mathbb{R}/\mathbb{Z}$ by identifying 2 points from $\mathbb{R}$ whenever they have an integer distance.

Exercise 13. The rotation of S^1 by an angle $2\pi\varphi$ is (in each of the above representations) given by $F(x) = (x + \varphi) \mod 1$. Show for the corresponding discrete dynamical system that:

a) F has a periodic orbit if and only if $\varphi \in \mathbb{Q}$, and in this case all orbits are periodic. (2 Points)

b) If $\varphi \notin \mathbb{Q}$ then every positive orbit $O(x) = \{F^n(x) : n = 0, 1, \ldots\}$ is dense in S^1. (6 Points)

Exercise 14. Show that the angular doubling map $F(x) = 2x$ (of $\mathbb{R}/\mathbb{Z} \cong S^1$) exhibits chaos on S^1. (10 Points)

Hint. Consider the binary expansion $x = 0.b_1b_2b_3\cdots \in [0,1]$ with $b_j \in \{0,1\}$. How does F act in this representation?

Exercise 15. Show for the Lotka-Volterra prey-predator system

$$\dot{x} = (\alpha - \beta y)x$$
$$\dot{y} = (\delta x - \gamma)y$$

(with fixed $\alpha, \beta, \gamma, \delta > 0$) that the critical point $(x_0, y_0) = (\gamma/\delta, \alpha/\beta)$ is stable. (6 Points)

Hint. Choose a Ljapunov function of the form $V(x,y) = F(x) + G(y)$, or recall an exercise from the previous semester.