7. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, December 7, 2016

Exercise 16. Let $u \in C(\Omega, \mathbb{R})$ be subharmonic. Show for every ball $B_\rho(x_0) \subseteq \Omega$ on which u is Lebesgue integrable (e.g. $K_\rho(x_0) \subseteq \Omega$) that

$$u(x_0) \le \frac{1}{\operatorname{mes} B_\rho(x_0)} \int_{B_\rho(x_0)} u(x)\,dx.$$

An analogous formula with $\ge$ holds for superharmonic functions. (4 Points)

Hint. Use without proof the Cavalieri principle in polar coordinates:

$$\int_{B_\rho(x_0)} u(x)\,dx = \int_0^\rho \int_{S_r(x_0)} u\,d\sigma\,dr.$$

Exercise 17. Let $u \in C^3(\Omega, \mathbb{R})$ be harmonic. Show successively:

a) Every partial derivative u_{x_i} is harmonic. (2 Points)

b) (Cauchy estimate for u') If $|u|$ is bounded on $B_r(x_0) \subseteq \Omega$ by M then

$$|u_{x_i}(x_0)| \le \frac{M}{\operatorname{mes} B_r(x_0)}. \qquad \text{(6 Points)}$$

Hint. Apply Exercise 16 and integration by parts for the product $u_{x_i} \cdot 1$ on a ball $B_\rho(x_0)$ with $\rho < r$.

c) (Liouville's theorem) If $\Omega = \mathbb{R}^n$ and u is (globally) bounded then u is constant. (4 Points)

Exercise 18. Let $\Omega \subseteq \mathbb{R}^n$ be open, and $x_0 \in \Omega$. Show that there is no function $\delta\colon \Omega \to [-\infty, \infty]$ such that

$$\int_\Omega \delta(x) u(x)\,dx = u(x_0)$$

for every $u \in C^\infty(\mathbb{R}^n, \mathbb{R})$. (The integral is understood in the sense of Lebesgue.) (4 Points)

Hint. Use without proof that there is $v \in C^\infty(\mathbb{R}^n, [0,1])$ satisfying $v(0) = 1$, which has its *support*

$$\operatorname{supp} v := \overline{\{x \in \mathbb{R}^n : v(x) \ne 0\}}$$

contained in $K_1(0)$, and consider $u_n(x) := v(n(x - x_0))$