8. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, December 14, 2016

Exercise 19. Solve the wave equation $u_{tt} = u_{xx}$ for $(t,x) \in [0,\infty) \times [0,1]$ with boundary condition $u(\cdot,0) = u(\cdot,1) = 0$ and initial conditions $u(0,\cdot) = 0$, $u_t(0,\cdot) = 1$. Describe (or sketch) the graph of $u|_{[0,1]^2}$ and $u|_{[1,2]\times[0,1]}$. (8 Points)

Exercise 20. Consider the assertion: There is a sequence of step functions $f_n\colon [0,1] \to \mathbb{R}$ satisfying $f_n(x) \to f(x)$ for all $x \in [0,1]$ and $\int_{[0,1]} |f_n(x)|\, dx \not\to \int_{[0,1]} |f(x)|\, dx$.
Give an example which shows that this assertion is true and verify that your example contradicts neither the definition of the integral nor the Lemma of Fatou nor Lebesgue's dominated convergence theorem. (6 Points)

Exercise 21. Show the following strong form of the variational lemma: Let $\Omega \subseteq \mathbb{R}^n$ be open and $u\colon \Omega \to \mathbb{R}$ be such that

$$\int_\Omega u\varphi\, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

(The assumption implies in particular that $u\varphi$ is Lebesgue integrable.) Show that $u = 0$ a.e. (6 Points)

Hint. Assume $u \neq 0$, without loss of generality $u(x) > 0$ for all x from a subset E of positive measure. There is a compact subset $K \subseteq E$ of positive measure (this follows from the so-called regularity of the Lebesgue measure; you do not have to prove this). Recall that by the lecture there is some $r > 0$ such that $K_0 := \overline{B_r(K)}$ is a compact subset of Ω. The hypothesis obviously implies that u is integrable on K_0.
Now consider $\varphi_k = g_k * \chi_K$ where g_k is a mollifier sequence, and use the Theorem of Riesz: If $\|f - f_k\|_{L_p} \to 0$ then there is a subsequence with $f_{k_j}(x) \to f(x)$ for almost all x.

Exercise 22. Let M be a subset of a Banach space with the poperty that there is an uncountable family $y_i \in M$ $(i \in I)$ with $\|y_i - y_j\| \geq 1$ for all $i \neq j$. Show that M is not separable. Conclude that the function $f\colon [0,1] \to L_p([0,1])$,

$$f(x) = \chi_{(0,x]}$$

fails to be measurable if $p = \infty$, although in case $1 \leq p < \infty$ the function f is even continuous. (4 + 4 Points)