9. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: (This week later than usual) Monday, January 9, 2017

Exercise 23. Show that if $f, g, h \in L_1(\mathbb{R}^n)$ then $f * g$ is almost everywhere defined and belongs to $L_1(\mathbb{R}^n)$ with $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$,

$$\int_{\mathbb{R}^n} (f * g)(x)\, dx = \int_{\mathbb{R}^n} f(x)\, dx \int_{\mathbb{R}^n} g(x)\, dx,$$

and $(f * g) * h = f * (g * h)$. (8 Points)

Exercise 24. Show the following generalization of Hölder's inequality:

$$\|f_1 \cdots f_n\|_q \leq \|f_1\|_{p_1} \cdots \|f_n\|_{p_n},$$

where $p_1, \ldots, p_n, q \in [1, \infty]$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = \frac{1}{q}$. (8 Points)

Exercise 25. Let $1 \leq q < p \leq \infty$.

a) Show that if $\operatorname{mes} \Omega < \infty$ then $L_p(\Omega) \subseteq L_q(\Omega)$, and the embedding operator is bounded: $\|x\|_q \leq C \|x\|_p$ where $C = (\operatorname{mes} \Omega)^{(1/q)-(1/p)}$. (4 Points)

Hint. Hölder's inequality.

b) Is one of the sets $L_p([0, \infty))$ and $L_q([0, \infty))$ a subset of the other? If yes, which is the larger one? (6 Points)