10. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, January 11, 2017

Exercise 26. Let I be a compact metric space, X a Banach space, Y a normed space, and $F\colon I \times X \to Y$ such that $U(t) := F(t,\cdot)\colon X \to Y$ is linear.
The map U is called *strongly continuous* if $U(t)u$ is a continuous function of t for every fixed $u \in X$.
We denote by $\mathfrak{L}(X,Y)$ the space of all bounded linear operators $A\colon X \to Y$, endowed with the operator norm.
Show that the following assertions are equivalent:

1. F is continuous (as a function of two variables).
2. F is continuous in each variable separately, that is, $F(t,u)$ is continuous for every *fixed* u as a function of t and for every *fixed* t as a function of u.
3. U is *strongly continuous*, and $U(t) \in \mathfrak{L}(X,Y)$ for every $t \in I$.
4. U is *strongly continuous*, and there is a constant $C < \infty$ such that $\|U(t)\| \leq C$ for all $t \in I$.

Is also the following property equivalent: $U\colon I \to \mathfrak{L}(X,Y)$ is continuous (in operator norm)? (8 + 8 Points)

Remark. If I is an interval (not necessarily compact), the first two assertions are still equivalent, because one can restrict considerations to a compact subinterval. The same can be shown (by using analogous proofs) if I is a normed space and F is linear in both variables.

Exercise 27. Let $X = C([0,1])$ (with the max-norm), and $J \in \mathfrak{L}(X)$ defined by $J(u) := \int_0^t u(s)\,ds$. With $Y := J(X)$, obviously $J\colon X \to Y$ is a bijection. Show that the inverse $D\colon Y \to X$, $Du := u'$ is unbounded. Why does this not contradict the inverse mapping theorem? (4 + 4 Points)