11. Exercise for Differential Equations II (WS 2016/17, Väth)
Time limit: Wednesday, January 18, 2017

We use the notation $\mathfrak{L}(X,Y) = \{A \mid A\colon X \to Y \text{ is linear and bounded}\}$ and $X^* := \mathfrak{L}(X,\mathbb{K})$.

Exercise 28. Let X and Y be normed space, Y being complete. Let $U \subseteq X$ be a dense subspace, i.e. $\overline{U} = X$. Show that any bounded linear operator $A \in \mathfrak{L}(U,Y)$ has a unique extension to a bounded linear operator $A \in \mathfrak{L}(X,Y)$. (6 Points)

Exercise 29. What is wrong with this argument:
Consider the Hilbert space $X = L_2([-1,1])$. On the dense subspace $U = C([-1,1])$ define the bounded linear functional $\ell\colon U \to \mathbb{K}$ by $\ell(u) := u(0)$ and extend it to X (Exercise 28).
By the Riesz representation theorem there is some $\delta \in X$ with

$$\ell(u) = \langle u, \delta \rangle = \int_{[-1,1]} \delta(x)u(x)\,dx \quad \text{for all } x \in X.$$

In particular, this holds for all $x \in C([-1,1])$, that is, there is indeed a *function* $\delta \in L_2([-1,1])$ such that

$$\int_\Omega \delta(x)u(x)\,dx = \ell(u) = u(0) \quad \text{if } u \in C([-1,1]).$$

(Of course, this is not possible, not even for $u \in C_c^\infty([-1,1])$ as you know from an earlier exercise; so there must be a mistake somewhere in this argument!) (6 Points)

Exercise 30. A sequence u_n in a normed space X is *weakly convergent to u* (in symbols: $u_n \rightharpoonup u$), if $\ell(u_n) \to \ell(u)$ for every bounded linear functional $\ell \in X^*$. Show for the case that X is a Hilbert space:

a) $u_n \to u$ implies $u_n \rightharpoonup u$, and the converse is false. (1 + 4 Points)
Hint. Consider a Hilbert space X with a countable orthonormal basis e_n and use Bessel's inequality.

b) If $u_n \rightharpoonup u$ then $\|u_n\|$ is bounded. (6 Points)
Hint. Consider the family $A_n \in \mathfrak{L}(X,\mathbb{K})$, $A_n v := \langle v, u_n \rangle$.

c) If $u_n \rightharpoonup u$ then $\|u\| \leq \liminf_{n\to\infty} \|u_n\|$. (4 Points)

d) $u_n \to u \iff (u_n \rightharpoonup u \text{ and } \|u_n\| \to \|u\|)$. (4 Points)

Remark. All assertions in this exercise except for the last hold also if X is not a Hilbert space, but the proofs require then the Theorem of Hahn-Banach (and some of its consequences) which we have not discussed.
The last property holds in some Banach spaces (e.g. in $L_p(\Omega)$ if $1 < p < \infty$) but not in all; it is called the Kadec-Klee property.