12. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, January 25, 2017

Exercise 31. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain. Show that for every $f \in L_2(\Omega)$ and $\lambda \le 0$ the boundary value problem

$$\begin{cases} -\Delta u = \lambda u + f & \text{on } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

has a unique weak solution $u \in W_0^{1,2}(\Omega)$, and that there is a constant C_λ which is independent of f such that $\|u\|_{W^{1,2}(\Omega)} \le C_\lambda \|f\|_{L_2(\Omega)}$. (8 Points)

Remark. Analogous assertions hold for $f \in L_p(\Omega)$ with $u \in W^{1,p}(\Omega)$, respectively, for arbitrary elliptic operators instead of Δ, and also if $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \le 0$. However, the proofs in the non-Hilbert space case $p \neq 2$ are much harder (and e.g. reduced to the Hilbert space case by approximation or regularity arguments).

Exercise 32. Show that if X is a separable Hilbert space then every bounded sequence $u_n \in X$ has a weakly convergent subsequence. (8 Points)

Remark. The separability is actually not necessary, because one can work in the separable subspace $X_0 = \overline{\operatorname{span}\{u_1, u_2, \ldots\}}$ and use that $X_0 = (X_0^\perp)^\perp$.

Remark. One can show that this assertion is true for a larger class of spaces than Hilbert spaces: It characterizes so-called *reflexive* spaces. The spaces $L_p(\Omega)$ and $W^{n,p}(\Omega)$ are reflexive if $1 < p < \infty$ (but not if $p = 1$ or $p = \infty$).

Hint. Let $\overline{\{v_1, v_2, \ldots\}} = X$.
Show by a diagonal argument that there is a subsequence u_{n_k} such that $\lim_{k\to\infty} \langle v_j, u_{n_k}\rangle = \ell_j$ exists for every $j = 1, 2, \ldots$
It follows that if $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$ then $\ell(v) := \lambda_1 \ell_1 + \cdots + \lambda_n \ell_n$ defines a linear functional ℓ on the subspace $U = \operatorname{span}\{v_1, v_2, \ldots\}$ such that $\ell(v) = \lim_{k\to\infty} \langle v, u_{n_k}\rangle$. Extend ℓ to some $\ell \in X^*$. By Riesz, there is u with $\ell(x) = \langle x, u\rangle$. Then $u_{n_k} \rightharpoonup u$.

Theorem (Rellich). *If $\Omega \subseteq \mathbb{R}^N$ is bounded and $1 \le q < p_*$ then $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ implies $u_n \to u$ in $L_q(\Omega)$. In case $p \ge N$, we put here $p_* = \infty$.*

Exercise 33. Show that Rellich's embedding theorem implies for $p = q = 2$ that the embedding $W_0^{1,p}(\Omega) \to L_q(\Omega)$ is compact, that is, the identity map $i\colon W_0^{1,p}(\Omega) \to L_q(\Omega)$ sends bounded sets into relatively compact sets. (4 Points)

Remark. Using the above remarks, this follows also in case $1 < p < \infty$ and $1 \le q < p_*$. More general, the Sobolev embeddings $i\colon W_0^{n,p}(\Omega) \to W^{m,q}(\Omega)$ and, if Ω is a bounded extension domain, $i\colon W^{n,p}(\Omega) \to W^{m,q}(\Omega)$ are compact when $n > m$ and, in case $1 \le p \le N$, additionally

$$n - \frac{N}{p} > m - \frac{N}{q}.$$

Show the case $p > N$ when $m = 0$, that is, show the compactness of the embedding $i\colon W^{n,p}(\Omega) \to L_q(\Omega)$ if $p > N$. (4 Points)

Hint. Note first that $C^\alpha(\Omega) = C^\alpha(\overline{\Omega})$ in a trivial way. Now equip the space $C^\alpha(\overline{\Omega})$ $(0 < \alpha < 1)$ of Hölder continuous functions with exponent α with the norm

$$\|u\| = \|u\|_{C(\overline{\Omega})} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha}.$$

Note that by definition of the supremum, the last summand is the smallest constant L such that

$$|u(x) - u(x)| \le L\,|x - y|^\alpha.$$

Use the Arzelà-Ascoli theorem to show that the embedding $C^\alpha(\Omega) \to C(\overline{\Omega})$ is compact. Now consider the chain of bounded embeddings (with $\alpha = 1 - \frac{N}{p}$)

$$W^{n,p}(\Omega) \subseteq W^{1,p}(\Omega) \subseteq C^\alpha(\Omega) \subseteq C(\overline{\Omega}) \subseteq L_\infty(\Omega) \subseteq L_q(\Omega).$$