13. Exercise for Differential Equations II (WS 2016/17, Väth)

Time limit: Wednesday, February 1, 2017

Exercise 34. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded Lipschitz domain. Let us interpret Δ together with the Dirichlet boundary condition as an operator A in $X = L_2(\Omega)$ in the weak sense, that is, we say that Au is defined and equal to $f \in X$ if $u \in W^{1,2}(\Omega)$ is a weak solution of

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u|_{\partial\Omega} = 0. \end{cases} \tag{$*$}$$

In particular, $A\colon D(A) \to X$, where the domain $D(A) \subseteq X$ consists of those functions u from $W_0^{1,2}(\Omega)$ for which there is some $f \in L_2(\Omega)$ such that $(*)$ holds in the weak sense.

Show by the previous exercises that every $\lambda \geq 0$ belongs to the resolvent set of A and that the corresponding resolvent $R_A(\lambda) = (\lambda I - A)^{-1} \in \mathfrak{L}(X)$ is compact in X, that is, it sends bounded sets into relatively compact sets. (8 Points)

Remark. Analogous assertions hold with $X = L_p(\Omega)$, for an arbitrary elliptic operators instead of Δ, and also if λ is complex with $\operatorname{Re}\lambda \leq 0$. Again, the proofs in case $p \neq 2$ are much harder.

Exercise 35. Show that the resolvent in Exercise 34 satisfies in the space $X = L_2(\Omega)$ an estimate of the form

$$\|R_A(\lambda)\|_{\mathfrak{L}(X)} \leq \frac{1}{\lambda} \quad \text{if } \lambda > 0.$$

Use this to show that A is the generator of a C_0 semigroup. (8 Points)

Hint. Multiply the equation $u = R_\lambda(A)f$ with $R_\lambda(A)^{-1}$, and then form the scalar product (in X) with u.

Remark. An analogous assertion can be shown for Neumann boundary conditions and even for mixed boundary conditions (with Dirichlet conditions on some part and Neumann conditions on the remaining part of the boundary $\partial\Omega$).

Exercise 36. Let X be a Banach space, and $D(A) \subseteq X$ a linear subspace. For $A\colon D(A) \to X$, we define the graph norm on $D(A)$ by $\|x\|_A = \|x\|_X + \|Ax\|_X$. Show:

a) $D(A)$ with the graph norm is a Banach space if and only if A is closed. (5 Points)
b) When we equip $Y = D(A)$ with the graph norm then $A\colon Y \to X$ is bounded. (1 Point)
c) The norms $\|\cdot\|_A$ and $\|\cdot\|_X$ are equivalent on $D(A)$ if and only if $A\colon D(A) \to X$ is bounded (as an operator in X). (2 Points)