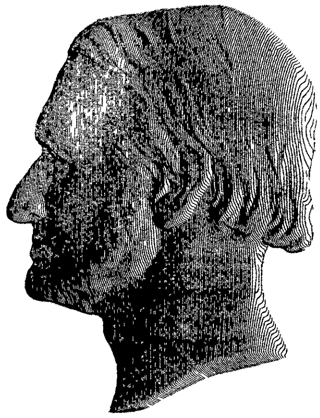
CARL FRIEDRICH GAUSS

WERKE

DRITTER BAND



HERAUSGEGEBEN

VON DER

KÖNIGLICHEN GESELLSCHAFT DER WISSENSCHAFTEN

ZU

GÖTTINGEN

1866.

NACHLASS.

[ARITHMETISCH GEOMETRISCHES MITTEL.]

PARS I.

DE ORIGINE PROPRIETATIBUSQUE GENERALIBUS NUMERORUM MEDIORUM ARITHM. GEOMETRICORUM.

1.

Sint $\begin{Bmatrix} a, a', a'', a'''\ldots \\ b, b', b'', b'''\ldots \end{Bmatrix}$ duae progressiones quantitatum ea lege formatae, ut quilibet ipsarum termini correspondentes sint *media* inter terminos antecedentes, et quidem termini progressionis superioris media arithmetica, progressionis inferioris geometrica, puta

$$a' = \tfrac{1}{2}(a+b),\ b' = \sqrt{ab},\ a'' = \tfrac{1}{2}(a'+b'),\ b'' = \sqrt{a'b'},\ a''' = \tfrac{1}{2}(a''+b''),\ b''' = \sqrt{a''b''} \text{ etc.}$$

Supponemus autem, ipsos a, b esse reales positivos, et pro radicibus quadraticis ubique accipi valores positivos; quo pacto progressiones quousque libuerit produci poterunt, omnes ipsarum termini erunt plene determinati valoresque positivos reales nanciscentur. Porro prima hic se fronte offerunt observationes sequentes:

I. Si $a = b$, omnes utriusque seriei termini erunt $= a = b$.

II. Si vero a, b sunt inaequales, erit $(a'-b')(a'+b') = \frac{1}{4}(a-b)^2$, unde concluditur $b' < a'$, et perinde erit $b'' < a''$, $b''' < a'''$ etc., i. e. quivis terminus seriei inferioris minor erit quam correspondens superioris. Quocirca in hoc casu supponemus, esse etiam $b < a$.

III. Eadem suppositione erit $a' < a$, $b' > b$; $a'' < a'$, $b'' > b'$ etc.; progressio itaque superior continuo decrescit, inferior continuo crescit; hinc manifestum est, utramque habere limitem; hi limites commode exprimuntur per a^∞, b^∞.

IV. Denique ex $\frac{a'-b'}{a-b} = \frac{(a-b)}{4(a'+b')} = \frac{a-b}{2(a+b)+4b'}$ sequitur $a'-b' < \frac{1}{2}(a-b)$, eodemque modo erit $a''-b'' < \frac{1}{2}(a'-b')$ etc. Hinc concluditur, $a-b$, $a'-b'$,

$a'' - b''$, $a''' - b'''$ etc. constituere progressionem continuo decrescentem atque ipsius limitem esse $= 0$. Hinc $a^\infty = b^\infty$, i. e. progressio superior et inferior eundem limitem habebunt, quo illa semper manet maior, haec minor.

Hunc limitem vocamus *numerum medium arithmetico-geometricum inter a et b*, et per $\mathrm{M}(a, b)$ designamus.

2.

Radices aequationis $xx - 2ax + bb = 0$ erunt reales positivi, siquidem $a \geqq b$; medium arithmeticum inter has radices erit a, geometricum b; designata itaque una radice (et quidem maiore si sunt inaequales) per $'a$, altera per $'b$, poterit $'a$ spectari tamquam terminus progressionis superioris terminum a praecedens, eodemque modo $'b$ tamquam terminus progressionis inferioris ante b. Similiter designando

aequationis $xx - 2\,'ax + {}'b\,'b = 0$	radicem maiorem per $''a$	minorem per $''b$
$xx - 2\,''ax + {}''b\,''b = 0$	$'''a$	$'''b$
$xx - 2\,'''ax + {}'''b\,'''b = 0$	$''''a$	$''''b$

poterunt $'a$, $''a$, $'''a$ etc. spectari tamquam continuatio progressionis superioris versus laevam, atque $'b$, $''b$, $'''b$ etc. tamquam continuatio progressionis inferioris, ita ut iam habeantur duae progressiones utrimque in infinitum continuabiles

$$\ldots\ldots {}''''a,\ {}'''a,\ {}''a,\ {}'a,\ a,\ a',\ a'',\ a''',\ a''''\ldots\ldots \tag{I}$$

$$\ldots\ldots {}''''b,\ {}'''b,\ {}''b,\ {}'b,\ b,\ b',\ b'',\ b''',\ b''''\ldots\ldots \tag{II}$$

Quivis itaque terminus progressionis (I) erit maior quam correspondens seriei (II); series illa a laeva ad dextram continuo decrescit, a dextra ad laevam crescit: haec a laeva ad dextram continuo crescit sensuque contrario decrescit. Versus dextram utraque series eundem limitem habet; versus laevam autem (I) super omnes limites crescit, (II) habet limitem 0 (nisi omnes utriusque progressionis termini sunt aequales). Nam $'a = a + \sqrt{(aa - bb)}$; $'b = a - \sqrt{(aa - bb)}$; hinc $'a'a - {}'b'b = 4a\sqrt{(aa - bb)} > 4(aa - bb)$, similiterque $''a''a - {}''b''b > 4('a'a - {}'b'b)$ etc., unde patet, seriem $aa - bb$, $'a'a - {}'b'b$, $''a''a - {}''b''b$ etc. et proin etiam hanc a, $'a$, $''a$ etc. quemvis limitem superare posse; $\frac{'b}{b} = \frac{b}{'a} < \frac{b}{a}$ similiterque $\frac{''b}{'b} < \frac{'b}{'a}$ etc. i. e. $\frac{^{n+1}b}{^{n}b}$ infra quemvis limitem deprimi potest augendo ipsum n, adeoque limes seriei b, $'b$, $''b$ etc. $= 0$.

Ex definitione numeri medii arithmetico-geometrici tam manifestum est, ut explicatione ampliore iam non opus sit, sequens

Theorema. *Numerus medius inter terminos quoscunque correspondentes progressionum I, II idem est atque inter a et b.*

3.

Quo clarius perspiciatur, quanta rapiditate series (I) et (II) ad dextram versus limitem suum approximent, et quomodo versus laevam illa crescat, haec decrescat, exempla quaedam hic sistimus:

Exemplum 1. $a = 1,\ b = 0,2$

${}''''a =$	15,83795 47919 02	${}''''b =$	0,00000 00000 00000 00005 7
${}'''a =$	7,91897 73959 512	${}'''b =$	0,00000 00013 481
${}''a =$	3,95948 86986 4971	${}''b =$	0,00010 30955 7682
${}'a =$	1,97979 58971 13271 23927 9	${}'b =$	0,02020 41028 86728 76072 1
$a =$	1,00000 00000 00000 00000 0	$b =$	0,20000 00000 00000 00000 0
$a' =$	0,60000 00000 00000 00000 0	$b' =$	0,44721 35954 99957 93928 2
$a'' =$	0,52360 67977 49978 96964 1	$b'' =$	0,51800 40128 22268 36005 0
$a''' =$	0,52080 54052 86123 66484 5	$b''' =$	0,52079 78709 39876 24344 0
$a'''' =$	0,52080 16381 12999 95414 3	$b'''' =$	0,52080 16380 99375
$a^{\mathrm{V}} =$	0,52080 16381 06187	$b^{\mathrm{V}} =$	0,52080 16381 06187

Hic a^{V}, b^{V} in 23^{a} demum figura discrepant; ${}^{\mathrm{V}}b$ est minor quam $(\frac{1}{10})^{40}$; ${}''''a$, ${}^{\mathrm{V}}a$, ${}^{\mathrm{VI}}a$ etc. sensibiliter formant progressionem geometricam, cuius exponens $= 2$.

Exemplum 2. $a = 1,\ b = 0,6.$

${}''''a =$	14,35538 2913	${}''''b =$	0,00000 00000
${}'''a =$	7,17769 14569 307	${}'''b =$	0,00001 73070
${}''a =$	3,58885 43819 99831 75712 7	${}''b =$	0,01114 56180 00168 24287 3
${}'a =$	1,80000 00000 00000 00000 0	${}'b =$	0,20000 00000 00000 00000 0
$a =$	1,00000 00000 00000 00000 0	$b =$	0,60000 00000 00000 00000 0
$a' =$	0,80000 00000 00000 00000 0	$b' =$	0,77459 66692 41483 37703 6
$a'' =$	0,78729 83346 20741 68851 8	$b'' =$	0,78719 58685 06172 16741 6
$a''' =$	0,78724 71015 63456 92796 7	$b''' =$	0,78724 70999
$a'''' =$	0,78724 71007 8	$b'''' =$	0,78724 71007 8

Exemplum 3. $a = 1,\ b = 0{,}8.$

${}^{\mathrm{v}}a$	$=25{,}19190\ 722$	${}^{\mathrm{v}}b$	$= 0{,}00000\ 00000\ 0$
${}''''a$	$=12{,}59595\ 36116\ 78$	${}''''b$	$= 0{,}00000\ 00133\ 367$
${}'''a$	$= 6{,}29797\ 68125\ 07655\ 42373\ 4$	${}'''b$	$= 0{,}00040\ 98644\ 58278\ 08440\ 9$
${}''a$	$= 3{,}14919\ 33384\ 82966\ 75407\ 2$	${}''b$	$= 0{,}05080\ 66615\ 17033\ 24592\ 8$
${}'a$	$= 1{,}60000\ 00000\ 00000\ 00000\ 0$	${}'b$	$= 0{,}40000\ 00000\ 00000\ 00000\ 0$
a	$= 1{,}00000\ 00000\ 00000\ 00000\ 0$	b	$= 0{,}80000\ 00000\ 00000\ 00000\ 0$
a'	$= 0{,}90000\ 00000\ 00000\ 00000\ 0$	b'	$= 0{,}89442\ 71909\ 99915\ 87856\ 4$
a''	$= 0{,}89721\ 35954\ 99957\ 93928\ 2$	b''	$= 0{,}89720\ 92687\ 32734$
a'''	$= 0{,}89721\ 14321\ 16346$	b'''	$= 0{,}89721\ 14321\ 13738$
a''''	$= 0{,}89721\ 14321\ 15042$	b''''	$= 0{,}89721\ 14321\ 15042$

Exemplum 4. $a = \sqrt{2},\ b = 1.$

${}''''a$	$=19{,}17024\ 37557\ 69475\ 31905\ 0$	${}''''b$	$= 0{,}00000\ 00009\ 32560\ 02627\ 6$
${}'''a$	$= 9{,}58512\ 18783\ 51017\ 67266\ 3$	${}'''b$	$= 0{,}00013\ 37064\ 06056\ 69181\ 0$
${}''a$	$= 4{,}79262\ 77923\ 78537\ 18223\ 7$	${}''b$	$= 0{,}03579\ 93323\ 67652\ 95745\ 7$
${}'a$	$= 2{,}41421\ 35623\ 73095\ 04880\ 2$	${}'b$	$= 0{,}41421\ 35623\ 73095\ 04880\ 2$
a	$= 1{,}41421\ 35623\ 73095\ 04880\ 2$	b	$= 1{,}00000\ 00000\ 00000\ 00000\ 0$
a'	$= 1{,}20710\ 67811\ 86547\ 52440\ 1$	b'	$= 1{,}18920\ 71150\ 02721\ 06671\ 7$
a''	$= 1{,}19815\ 69480\ 94634\ 29555\ 9$	b''	$= 1{,}19812\ 35214\ 93120\ 12260\ 7$
a'''	$= 1{,}19814\ 02347\ 93877\ 20908\ 3$	b'''	$= 1{,}19814\ 02346\ 77307\ 20579\ 8$
a''''	$= 1{,}19814\ 02347\ 35592\ 20744\ 1$	b''''	$= 1{,}19814\ 02347\ 35592\ 20743\ 9$

4.

Habeantur praeter progressiones

$$\ldots {}'''a,\quad {}''a,\quad {}'a,\quad a,\quad a',\quad a'',\quad a'''\ldots$$
$$\ldots {}'''b,\quad {}''b,\quad {}'b,\quad b,\quad b',\quad b'',\quad b'''\ldots$$

duae aliae

$$\ldots {}'''c,\quad {}''c,\quad {}'c,\quad c,\quad c',\quad c'',\quad c'''\ldots$$
$$\ldots {}'''d,\quad {}''d,\quad {}'d,\quad d,\quad d',\quad d'',\quad d'''\ldots$$

simili modo formatae; supponamusque, duos terminos in posterioribus duobus in prioribus *proportionales* esse, e. g. $a:b = c:d$, sive $a:c = b:d = 1:n$. Tunc omnes termini in I ad terminos in III, omnesque in II ad terminos in IV (similiter relative ad a, b, c, d siti ac sitos) in eadem ratione erunt, puta

$$c' = na', \quad c'' = na'', \quad c''' = na''', \quad 'c = 'an, \quad d^{\mathrm{V}} = nb^{\mathrm{V}}, \quad {}^{\mathrm{V}}d = {}^{\mathrm{V}}bn$$

Hinc facile deducitur, etiam limitem serierum I, II, fore ad limitem serierum III, IV ut 1 ad n, sive generaliter $\mathrm{M}(na, nb) = n\mathrm{M}(a, b)$. Erit itaque generaliter $\mathrm{M}(a, b) = a\mathrm{M}(1, \frac{b}{a}) = b\mathrm{M}(\frac{a}{b}, 1)$.

5.

Problema. *Exprimere medium arithmetico-geometricum inter numerum unitate maiorem $1+x$ et unitatem, per seriem secundum potestates ipsius x progredientem.*

Sol. Quum $\mathrm{M}(1, 1) = 1$, supponamus

$$\mathrm{M}(1+x, 1) = 1 + h'x + h''x^2 + h'''x^3 + h''''x^4 + \text{etc.}$$

ita ut h', h'', h''', h'''' sint coëfficientes constantes ab x non pendentes. Sit $x = 2t + tt$, eritque

$$\mathrm{M}(1+x, 1) = \mathrm{M}(1 + t + \tfrac{1}{2}tt,\ 1+t) = (1+t)\mathrm{M}(1 + \frac{\frac{1}{2}tt}{1+t}, 1).$$

Quare habebitur

$$\begin{aligned} &1 + h'(2t+tt) + h''(2t+tt)^2 + h'''(2t+tt)^3 + \text{etc.} \\ &= 1 + t + h'(\tfrac{1}{2}tt) + h''\frac{\frac{1}{4}t^4}{1+t} + h'''\frac{\frac{1}{8}t^6}{(1+t)^2} + \text{etc.} \end{aligned}$$

Hinc prodeunt aequationes

$$\begin{aligned} &2h' = 1 \\ &4h'' + h' = \tfrac{1}{2}h' \\ &8h''' + 4h'' = 0 \\ &16h'''' + 12h''' + h'' = \tfrac{1}{4}h'' \\ &32h^{\mathrm{V}} + 32h'''' + 6h''' = -\tfrac{1}{4}h'' \\ &64h^{\mathrm{VI}} + 80h^{\mathrm{V}} + 24h'''' + h''' = \tfrac{1}{4}h'' + \tfrac{1}{8}h''' \\ &128h^{\mathrm{VII}} + 192h^{\mathrm{VI}} + 80h^{\mathrm{V}} + 8h'''' = -\tfrac{1}{4}h'' - \tfrac{3}{8}h''' \\ &256h^{\mathrm{VIII}} + 448h^{\mathrm{VII}} + 240h^{\mathrm{VI}} + 40h^{\mathrm{V}} + h'''' = \tfrac{1}{4}h'' + \tfrac{3}{8}h''' + \tfrac{1}{16}h'''' \end{aligned}$$

etc. unde fit

$$h' = \tfrac{1}{2}, \quad h'' = -\tfrac{1}{16}, \quad h''' = \tfrac{1}{32}, \quad h'''' = -\tfrac{21}{1024}, \quad h^{\mathrm{V}} = \tfrac{31}{2048}, \quad h^{\mathrm{VI}} = -\tfrac{195}{16384}$$

Quare

$$\mathrm{M}(1+x, 1) = 1 + \tfrac{1}{2}x - \tfrac{1}{16}xx + \tfrac{1}{32}x^3 - \tfrac{21}{1024}x^4 + \tfrac{31}{2048}x^5 - \tfrac{195}{16384}x^6 \text{ etc.}$$

Ceterum nullo negotio perspicitur, medium inter 1 et numerum unitate minorem

$1-x$ fore $1-h'x+h''xx-h'''x^3+\text{etc.} = 1-\frac{1}{2}x-\frac{1}{16}xx-\frac{1}{32}x^3-\frac{21}{1024}x^4-\text{etc.}$ Quum hi coëfficientes legem obviam non exhibeant, has series praetergredimur, aliamque viam tentamus, quae successum feliciorem praestabit.

6.

Problema. *Exprimere medium ar. g. inter* $1+x$ *et* $1-x$ *per seriem secundum potestates ipsius* x *progredientem.*

Sol. Quum habeatur

$$M(1+x,\ 1-x) = (1-x)M(1+\frac{2x}{1-x},\ 1)$$

statim habetur e serie art. praec. substituendo ibi $\frac{2x}{1-x}$ pro x

$$M(1+\frac{2x}{1-x},\ 1) = 1+x+\frac{3}{4}xx+\frac{3}{4}x^3+\frac{43}{64}x^4+\frac{43}{64}x^5+\frac{161}{256}x^6+\text{etc.}$$

atque hinc

$$M(1+x,\ 1-x) = 1-\frac{1}{4}xx-\frac{5}{64}x^4-\frac{11}{256}x^6+\text{etc.}$$

Coëfficientes huius seriei etiam independenter a serie art. praec. per methodum sequentem erui possunt. Ponatur $x=\frac{2t}{1+tt}$ eritque

$$M(1+x,\ 1-x) = M(\frac{1-tt}{1+tt},\ 1) = \frac{1}{1+tt}M(1+tt,\ 1-tt)$$

Quare statuendo

$$M(1+x,\ 1-x) = 1+\alpha xx+\beta x^4+\gamma x^6+\delta x^8+\text{etc.}$$

(nam potestates ipsius x cum exponente impari non adesse sponte patet) habebitur

$$(1+tt)\{1+\alpha(\frac{2t}{1+tt})^2+\beta(\frac{2t}{1+tt})^4+\gamma(\frac{2t}{1+tt})^6+\delta(\frac{2t}{1+tt})^8+\text{etc.}\}$$
$$= 1+\alpha t^4+\beta t^8+\gamma t^{12}+\delta t^{16}+\text{etc.}$$

Hinc prodeunt aequationes

$1+4\alpha=0$	unde $\alpha=-\frac{1}{4}$
$-4\alpha+16\beta=\alpha$	$\beta=-\frac{5}{64}$
$4\alpha-48\beta+64\gamma=0$	$\gamma=-\frac{11}{256}$
$-4\alpha+96\beta-320\gamma+256\delta=\beta$	$\delta=-\frac{29}{1024}-\frac{5}{16384}=-\frac{469}{16384}$
etc.	etc.

In hac quoque serie coëfficientes legi simplici non subiecti sunt: at si unitas per illam seriem dividitur, prodit

$$\frac{1}{\mathrm{M}(1+x,\, 1-x)} = 1+\tfrac{1}{4}xx+\tfrac{9}{64}x^4+\tfrac{25}{256}x^6+\tfrac{1225}{16384}x^8+\text{ etc.}$$

ubi primo aspectu videmus, coëfficientes esse quadrata radicum $\frac{1}{2}$, $\frac{1}{2}.\frac{3}{4}$, $\frac{1}{2}.\frac{3}{4}.\frac{5}{6}$, $\frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{7}{8}$, adeoque secundum legem persimplicem progredi. Sed methodus, per quam ad hanc conclusionem pulcherrimam pervenimus, inductionis tantummodo vim habet; quam ad certitudinis gradum evehere in disquisitionibus sqq. nobis proponimus.

7.

Supponendo

$$\frac{1}{\mathrm{M}(1+x,\, 1-x)} = 1+Axx+Bx^4+Cx^6+\text{ etc.}$$

atque ut in art. praec. $x = \frac{2t}{1+tt}$ habemus

$$1+A\left(\frac{2t}{1+tt}\right)^2+B\left(\frac{2t}{1+tt}\right)^4+C\left(\frac{2t}{1+tt}\right)^6+D\left(\frac{2t}{1+tt}\right)^8+\text{ etc.}$$
$$= 1+tt+At^4+At^6+Bt^8+Bt^{10}+Ct^{12}+Ct^{14}+Dt^{16}+Dt^{18}+\text{ etc.}$$

unde emergunt aequationes

$$\begin{aligned} 4A &= 1 \\ -\,8A+\ 16B &= A \\ 12A-\ 64B+\ 64C &= A \\ -16A+160B-384C+256D &= B \quad \text{etc.} \end{aligned}$$

atque hinc $A=\frac{1}{4}$, $B=\frac{1}{4}.\frac{9}{16}$, $C=\frac{1}{4}.\frac{9}{16}.\frac{25}{36}$, $D=\frac{1}{4}.\frac{9}{16}.\frac{25}{36}.\frac{49}{64}$, ut supra: sed legis ratio hinc operosius deduceretur: quare methodum sequentem praeferimus. Ex aequatione

$$\frac{2t}{1+tt}+A\left(\frac{2t}{1+tt}\right)^3+B\left(\frac{2t}{1+tt}\right)^5+\text{ etc.} = 2t(1+At^4+Bt^8\ldots)$$

demanant aequationes sequentes:

$$1 = 1 \qquad [1]$$
$$0 = 1-\ 4A \qquad [2]$$
$$A = 1-12A+\ 16B \qquad [3]$$
$$0 = 1-24A+\ 80B-\ \ 64C \qquad [4]$$
$$B = 1-40A+240B-\ 448C+\ 256D \qquad [5]$$
$$0 = 1-60A+560B-1792C+2304D-1024E \qquad [6]$$

ubi coëfficientes facile subiiciuntur formulae generali: scilicet aequatio n^{ta} erit

$$M = 1 - 4A \times \frac{n.n-1}{1.2} + 16B \times \frac{n+1.n.n-1.n-2}{1.2.3.4} - 64\,C \times \frac{n+2.n+1.n.n-1.n-2.n-3}{1.2.3.4.5.6}$$
$$+ 256\,D \times \frac{n+3.n+2.n+1.n.n-1.n-2.n-3.n-4}{1.2.3.4.5.6.7.8} - \text{etc.}$$

ubi M erit vel $= 0$ (quando n par), vel aequalis termino $\frac{1}{2}(n+1)^{\text{to}}$ seriei $1, A, B, C, D$ etc. (quando n impar). Iam ex his aequationibus sequentes novas deducimus*).

$[2]$,	$0 =$	$1 - 4A$
$4[3] - [1]$,	$4A - 1 =$	$3 - 48A + 64B$
$9[4] - 4[2]$,	$0 =$	$5 - 200A + 720B - 576C$
$16[5] - 9[3]$,	$16B - 9A =$	$7 - 532A + 3696B - 7168C + 4096D$
$25[6] - 16[4]$,	$0 =$	$9 - 1116A + 12720B - 43776C + 57600D - 25600E$

etc., ubi coëfficientes legi generali facile subiiciuntur. Scilicet aequatio n^{ta} erit

$$nnN - (n-1)^2 L = (2n-1)\Big(1 - 4A \times \frac{3nn-3n+2}{1.2} + 16B \times \frac{n.n-1.5nn-5n+6}{1.2.3.4}$$
$$- 16\,C \times \frac{n+1.n.n-1.n-2.7nn-7n+12}{1.2.3.4.5.6}$$
$$+ 64D \times \frac{n+2.n+1.n.n-1.n-2.n-3.9nn-9n+20}{1.2.3.4.5.6.7.8} - \text{etc.}\Big) \;**)$$

ubi factorum progressio obvia est (puta praeter factores simplices in singulos coëfficientes ingreditur factor duplex talis $knn - kn + \frac{1}{4}(kk-1)$). Hae aequationes simplicius sequenti modo exhibentur, singularum membris ad dextram in binas partes discerptis (praeter aequ. primam, quae immutata retinetur):

$$\begin{aligned}
0 &= 1 - 4A \\
4A - 1 &= \begin{cases} 3 - 36A \\ \quad - 12A + 64B \end{cases} \\
0 &= \begin{cases} 5 - 180A + 400B \\ \quad - 20A + 320B - 576C \end{cases} \\
16B - 9A &= \begin{cases} 7 - 504A + 2800B - 3136C \\ \quad - 28A + 896B - 4032C + 4096D \end{cases} \\
0 &= \begin{cases} 9 - 1080A + 10800B - 28224C + 20736D \\ \quad - 36A + 1920B - 15552C + 36864D - 25600E \end{cases}
\end{aligned}$$

*) Signa derivationis explicantur in *Disquisitionibus Arithmeticis* art. 162.

**) L et N hic sunt vel utraque $= 0$ (quando n impar), vel resp. terminis $\frac{1}{2}n^{\text{tis}}$, $\frac{1}{2}n+1^{\text{tis}}$ seriei $1, A, B, C, D, E$ etc. aequales.

scilicet generaliter aequatio n^{ta}

$$nnN-(n-1)^2L$$

$$=(2n-1)\left\{\begin{array}{l} 1-3.4A\frac{n.n-1}{1\,.\,2}+5.16B\frac{n+1.n.n-1.n-2}{1\,.\,2\,.\,3\,.\,4}-7.64C\frac{n+2.n+1\ldots n-3}{1\,.\,2\ldots 6} \\ -\quad 4A\frac{2}{1.2}\quad+\quad 16B\frac{n.n-1.16}{1.2.3.4}\quad-\quad 64C\frac{n+1.n.n-1.n-2.54}{1.2.3.4.5.6} \\ \ldots\ldots\pm k.2^{k-1}K\,\frac{n+\frac{k-3}{2}.n+\frac{k-5}{2}.n+\frac{k-7}{2}\ldots n-\frac{k-1}{2}}{1\,.\,2\,.\,3\,.\,.\,.\,.\,k-1}\ldots \\ \ldots\ldots\pm\quad 2^{k-1}K\,\frac{n+\frac{k-5}{2}.n+\frac{k-7}{2}\ldots.n-\frac{k-3}{2}.\frac{1}{4}(k-1)^2}{1\,.\,2\ldots\ldots\ldots k-1}\ldots \end{array}\right\}$$

designante k indefinite quemvis imparem, sint ita

$$\begin{aligned} 0&=1-4A \\ 4A-1&=3(1-4A)-4(9A-16B) \\ 0&=5(1-4A)-20(9A-16B)+16(25B-36C) \\ 16B-9A&=7(1-4A)-56(9A-16B)+112(25B-36C)-64(49C-64D) \\ 0&=9(1-4A)-120(9A-16B)+432(25B-36C)-576(49C-64D) \\ &\qquad+256(81D-100E) \end{aligned}$$

etc. et generaliter aequ. n^{ta}

$$nnN-(n-1)^2L$$

$$=(2n-1)(1-4A)-4\frac{2n-1.n.n-1}{1\,.\,2\,.\,3}(9A-16B)+16\frac{2n-1.n+1.n.n-1.n-2}{1\,.\,2\,.\,3\,.\,4\,.\,5}(25B-36C)$$

$$-64\frac{2n-1.n+2.n+1.n.n-1.n-2.n-3}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7}(49C-64D)+\text{ etc.}$$

ubi legis obviae universalitas e calculo sponte demanat. Hinc vero perspicuum est, fieri necessario

$$0=1-4A,\ 0=9A-16B,\ 0=25B-36C,\ 0=49C-64D,\ 0=81D-100E$$

etc. in inf. adeoque

$$A=\tfrac{1}{4},\ B=\tfrac{1}{4}\cdot\tfrac{9}{16},\ C=\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36},\ D=\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36}\cdot\tfrac{49}{64},\ E=\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36}\cdot\tfrac{49}{64}\cdot\tfrac{81}{100}$$

et sic porro in infinitum. Q. E. D.

8.

Si statuimus

$$1+\tfrac{1}{4}xx+\tfrac{1}{4}\cdot\tfrac{9}{16}x^4+\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36}x^6+\text{etc.}=y$$

fit

$$\tfrac{1}{2}xx+\tfrac{1}{4}\cdot\tfrac{9}{4}x^4+\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{6}x^6+\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36}\cdot\tfrac{49}{8}x^8+\text{ etc.}=\frac{x\,\mathrm{d}y}{\mathrm{d}x}$$

atque

$$xx+\tfrac{1}{4}\cdot 9x^4+\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot 25x^6+\tfrac{1}{4}\cdot\tfrac{9}{16}\cdot\tfrac{25}{36}\cdot 49x^8+\text{ etc.}=\frac{xx\,\mathrm{dd}y}{\mathrm{d}x^2}+\frac{x\,\mathrm{d}y}{\mathrm{d}x}$$

unde sponte sequitur

$$\frac{xx\,\mathrm{dd}y}{\mathrm{d}x^2}+3\frac{x\,\mathrm{d}y}{\mathrm{d}x}+y=\frac{1}{xx}\left(xx\frac{\mathrm{dd}y}{\mathrm{d}x^2}+\frac{x\,\mathrm{d}y}{\mathrm{d}x}\right)$$

sive

$$(x^3-x)\frac{\mathrm{dd}y}{\mathrm{d}x^2}+(3xx-1)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=0$$

Hoc itaque modo media nostra arithmetico-grometrica ad quantitates integrales revocata sunt, solutionemque particularem huiusce aequationis differentio-differentialis subministrant.

Eiusdem aequationis int. compl. est $\frac{\mathfrak{A}}{M(1+x,\ 1-x)}+\frac{\mathfrak{B}}{M(1,x)}$.

Sit φ angulus indefinitus, eritque valor integralis $\int\cos\varphi^2\,\mathrm{d}\varphi$, a $\varphi=0$ usque ad $\varphi=\pi$, ut vulgo notum est, $=\frac{1}{2}\pi$; eodem modo fit valor integralis $\int\cos\varphi^4\,\mathrm{d}\varphi$ inter eosdem limites $=\frac{1}{2}\cdot\frac{3}{4}\pi$; valor integralis $\int\cos\varphi^6\,\mathrm{d}\varphi=\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\pi$ etc. — denique, ut sponte patet, $\int\mathrm{d}\varphi=\pi$. Hinc perspicuum est, valorem integralis

$$\int\mathrm{d}\varphi\times(1+\tfrac{1}{2}x^2\cos\varphi^2+\tfrac{1}{2}\cdot\tfrac{3}{4}x^4\cos\varphi^4+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}x^6\cos\varphi^6+\text{ etc.})$$

sive huius $\int\frac{\mathrm{d}\varphi}{\sqrt{(1-xx\cos\varphi^2)}}$, fieri $=\pi y$, si sumatur a $\varphi=0$ usque ad $\varphi=\pi$, spectando quantitatem x tamquam constantem.

Quodsi functio $\frac{1}{\sqrt{(1-xx\cos\varphi^2)}}$ in seriem talem evolvi supponatur,

$$P+2Q\cos 2\varphi+2R\cos 4\varphi+2S\cos 6\varphi+\text{ etc.}$$

ita ut coëfficientes P, Q, R, S etc. a sola x pendeant: valor integralis supra traditi completus erit

$$P\varphi+Q\sin 2\varphi+\tfrac{1}{2}R\sin 4\varphi+\tfrac{1}{3}S\sin 6\varphi+\text{ etc. }+\text{Const.}$$

adeoque valor intra limites ante allatos, $= P\varphi$, unde $y = P$. Iam observamus, quum sit $\frac{1}{y} = \mathrm{M}(1+x, 1-x)$ fieri quoque $\frac{1}{y} = \mathrm{M}(1, \sqrt{(1-xx)})$. Hinc facillime derivatur sequens theorema generalius: Si expressio talis $\frac{\alpha}{\sqrt{(6-\gamma\cos\varphi^2)}} = W$ in seriem secundum cosinus angulorum 2φ, 4φ, 6φ etc. progredientem evolvatur, cuius terminus constans designetur per P; valores maximus et minimus ipsius W autem (puta $\frac{\alpha}{\sqrt{(6-\gamma)}}$ et $\frac{\alpha}{\sqrt{6}}$) denotentur per v, v': erit $\frac{1}{P}$ medium arithmetico-geometricum inter $\frac{1}{v}$ et $\frac{1}{v'}$. Proxime quidem ratiocinia praecedentia pro eo tantummodo casu valent, ubi γ est quantitas positiva: sed facillime ad eum quoque casum extenduntur, ubi γ est negativus. In hocce enim casu fit $W = \frac{\alpha}{\sqrt{(6-\gamma+\gamma\cos\psi^2)}}$ ponendo $\psi = 90-\varphi$; unde $\frac{1}{P}$ med. inter $\frac{\sqrt{6}}{\alpha}$ et $\frac{\sqrt{(6-\gamma)}}{\alpha}$. Ut supra. Ceterum nullo negotio patet, idem theorema sine ulla mutatione etiam ad expressiones tales $\frac{\alpha}{\sqrt{(6+\gamma\cos\varphi)}}$ extendi, puta ut $\frac{1}{\text{Term. Const.}}$ fiat

$$= \mathrm{M}\left(\frac{1}{\text{valor max.}}, \frac{1}{\text{valor minim.}}\right) = \mathrm{M}\left(\frac{\sqrt{(6-\gamma)}}{\alpha}, \frac{\sqrt{(6+\gamma)}}{\alpha}\right)$$

huiusmodi enim functiones reducentur ad formam $\frac{\alpha}{\sqrt{(6-\gamma+2\gamma\cos\psi^2)}}$, ei de qua modo diximus prorsus similem, si scribatur $\varphi = 2\psi$ *).

Denique monemus, in sequentibus demonstrationem multo generaliorem eorundem theorematum ex principiis magis genuinis datum iri; praetereaque mox etiam omnes reliquos coëfficientes Q, R, S etc. per methodos aeque expeditas eruere docebimus. Hoc loco haec disquisitio eam quoque utilitatem praestabit, ut iis, qui dum veritatum aeternarum sublimitatem atque divinam venustatem non sapiunt, earum pretium ex solo usu, qui inde in partes matheseos applicatae redundare potest aestimare noverunt, — hasce investigationes cariores reddat. Quantae enim utilitatis sit evolutio coëfficientium P, Q, R, S etc. tam rapida, ut ea quae ex his principiis promanat, in astronomia physica sive theoria perturbationum planetarum, nemo ignorat.

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*) Ceterum ex praecedd. sponte sequitur, si plures expressiones tales $\frac{\alpha}{\sqrt{(6+\gamma\cos\varphi)}}$ ita comparatae sint, ut ipsarum valores extremi aequales sint: terminos constantes ex ipsarum evolutione prodeuntes necessario aequales fieri, etiamsi omnes reliqui coëfficientes valde discrepent.