Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 5: Consider the vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\dot{x} \;=\; \begin{pmatrix} a & b \\ -b & a \end{pmatrix} x,$$

with $a, b \in \mathbb{R}$. Transform this linear differential equation into polar coordinates:

$$x \;=\; \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix},$$

with $r > 0$, $\phi \in \mathbb{R}/2\pi\mathbb{Z}$. Choose $b \neq 0$ arbitrarily and sketch phase portraits in (r, ϕ)-coordinates and in x-coordinates for $a < 0$, $a = 0$, $a > 0$.

Problem 6: Let $\Phi_{t,s} : \mathbb{R}^N \to \mathbb{R}^N$ be a *periodic* evolution with period $p > 0$, i.e.

$$\text{for all } t, s \in \mathbb{R} : \quad \Phi_{t+p,s+p} = \Phi_{t,s}.$$

Consider the *stroboscope* map $\Pi : \mathbb{R}^N \to \mathbb{R}^N$,

$$\Pi(x) = \Phi_{p,0}(x).$$

Prove:

(i) for all $k \in \mathbb{N}$: $\Phi_{kp,0} = \Pi^k$;

(ii) for each $t \in \mathbb{R}$ there exists a change of coordinates $\psi : \mathbb{R}^N \to \mathbb{R}^N$ such that for all $k \in \mathbb{Z}$: $\Phi_{t+kp,t} = \psi\Pi^k\psi^{-1}$. Determine ψ.

Problem 7: Consider the map

$$\Phi_t(x_1, x_2) = (x_1 + t, x_2 + \sigma t), \qquad \sigma \in \mathbb{R},$$

and observe that

$$\Phi_t(x_1 + k, x_2 + n) = \Phi_t(x_1, x_2) + (k, n), \qquad \forall k, n \in \mathbb{Z}.$$

Show that $\Phi_t(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}^2$ defines a flow on the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let σ be rational. How do typical trajectories look like? Describe the α- and ω-limits.

Extra credit: What happens if σ is irrational ?

Problem 8: Consider free fall through the center of mother earth. Smart people claim that, assuming constant density, the distance $r(t)$ from the center of the earth satisfies

$$\ddot{r} = -Dr,$$

for some suitable constant D. What is the period of the complete round trip starting at the surface of the earth. With relevant data from the internet compare your result with the results for Moon, Mars, Jupiter and the Sun.