Homework Assignments

Dynamical Systems I

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 9: Prove that area preserving planar flows are Hamiltonian.

More precisely, let $\Phi_t(x)$ denote a flow with associated C^1-vector field $\dot{x} = f(x)$, for planar $x \in \mathbb{R}^2$. Assume that Φ preserves two-dimensional Lebesgue measure, i.e. for any real t and any bounded open set $U \subset \mathbb{R}^2$ we have $\text{area}(\Phi_t(U)) = \text{area}(U)$.

(i) Prove $\det \partial_x \Phi_t(x) = 1$, for all t, x.

(ii) Calculate $\partial_t \det \partial_x \Phi_t(x)$, at $t = 0$.

(iii) Conclude that the vector field $f(x)$ is Hamiltonian.

Problem 10: [Arnol'd, 2.4.5] Determine all $k \in \mathbb{R}$ for which the planar system

$$\begin{aligned} \dot{x_1} &= x_1, \\ \dot{x_2} &= kx_2, \end{aligned}$$

possesses a nontrivial first integral $I(x_1, x_2)$ such that (i) I is C^0; (ii) I is C^1; (iii) I is regular. Consider each of the three cases separately.

Problem 11: Suppose that a flow on $X = \mathbb{R}^N$ possesses a periodic orbit Γ and an open neighborhood U of Γ such that every solution $x(t)$ that starts in U converges to the set Γ in forward time $t \to +\infty$.

Prove or disprove: Every continuous first integral I on X must be constant on U.

Problem 12: Suppose that a planar flow possesses a first integral $I(x)$, $x \in \mathbb{R}^2$, given by the nondegenerate quadratic form

$$I(x) := x^T A x, \qquad A = A^T, \ \det A \neq 0.$$

Prove or disprove: The origin $x = 0$ is an equilibrium.
Extra credit: How about the general case $x \in \mathbb{R}^N$?