Homework Assignments

Dynamical Systems I

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`http://dynamics.mi.fu-berlin.de/lectures/`

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Problem 17: Taylor expansion of the nonlinearity $\sin x$ in the mathematical pendulum leads to the cubic pendulum $\ddot{x} + x - x^3/6 = 0$.

Sketch the trajectories and discuss all solution types. Compare your results with the Duffing oscillator discussed in class. (It is admissible, but not required, to submit numerical plots of the phase plane. Such plots, however, cannot serve as *proofs* of your claims.)

Problem 18: *Morse Lemma.* Let $V \in C^3(\mathbb{R}, \mathbb{R})$ satisfy $V(0) = V'(0) = 0$ and $V''(0) \neq 0$. Show that there exists a local C^1 diffeomorphism $y = y(x)$, near $x = y = 0$, such that $V(x) = \operatorname{sign}(V''(0))\, y(x)^2/2$.

Now consider the pendulum $\ddot{x} + V'(x) = 0$ with potential energy V. Indicate how the coordinates $(y, \dot{x})$ provide hyperbola and separatrix trajectories near local maxima of V with $V'' < 0$.

Problem 19: Consider the pendulum equation

$$\ddot{x} + g(x) = 0$$

for an odd C^1 function $g : \mathbb{R} \to \mathbb{R}$, i.e. $g(-x) = -g(x)$ for all $x \in \mathbb{R}$. Assume $g(x) \cdot x > 0$ for all $x \neq 0$. Let $p(g, a) > 0$ be the minimal period of the solution with initial value $x(0) = a > 0,\ \dot{x}(0) = 0$.

Prove:

(i) If $g_1(x) < g_2(x)$ for all $x > 0$, then $p(g_1, a) > p(g_2, a)$ for all $a > 0$.

(ii) [Hard spring] If $x \mapsto g(x)/x$ is strictly monotonically increasing for $x > 0$, then $a \mapsto p(g, a)$ is strictly monotonically decreasing for $a > 0$.

Hint: $y(t) := \frac{a_1}{a_2} x(t)$ solves the equation $\ddot{y} + \tilde{g}(y) = 0$ with $\tilde{g}(y) := \frac{a_1}{a_2} g(\frac{a_2}{a_1} y)$.

Problem 20: Consider the constantly forced pendulum equation

$$\ddot{x} + \sin x = \varepsilon, \qquad x \in \mathbb{R}\,.$$

Plot the trajectories of the solutions for small $\varepsilon > 0$. Compare your results with the usual pendulum $\varepsilon = 0$. Explain the arising discrepancies. (Once again: it is admissible, but not required, to submit numerical plots of the phase plane. Such plots, however, cannot serve as *proofs* of your claims.)